MATHEMATICS

STUDENT'S TEXTBOOK GRADE 11

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FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA
MINISTRY OF EDUCATION

MATHEMATICS STUDENT'S TEXTBOOK GRADE 11

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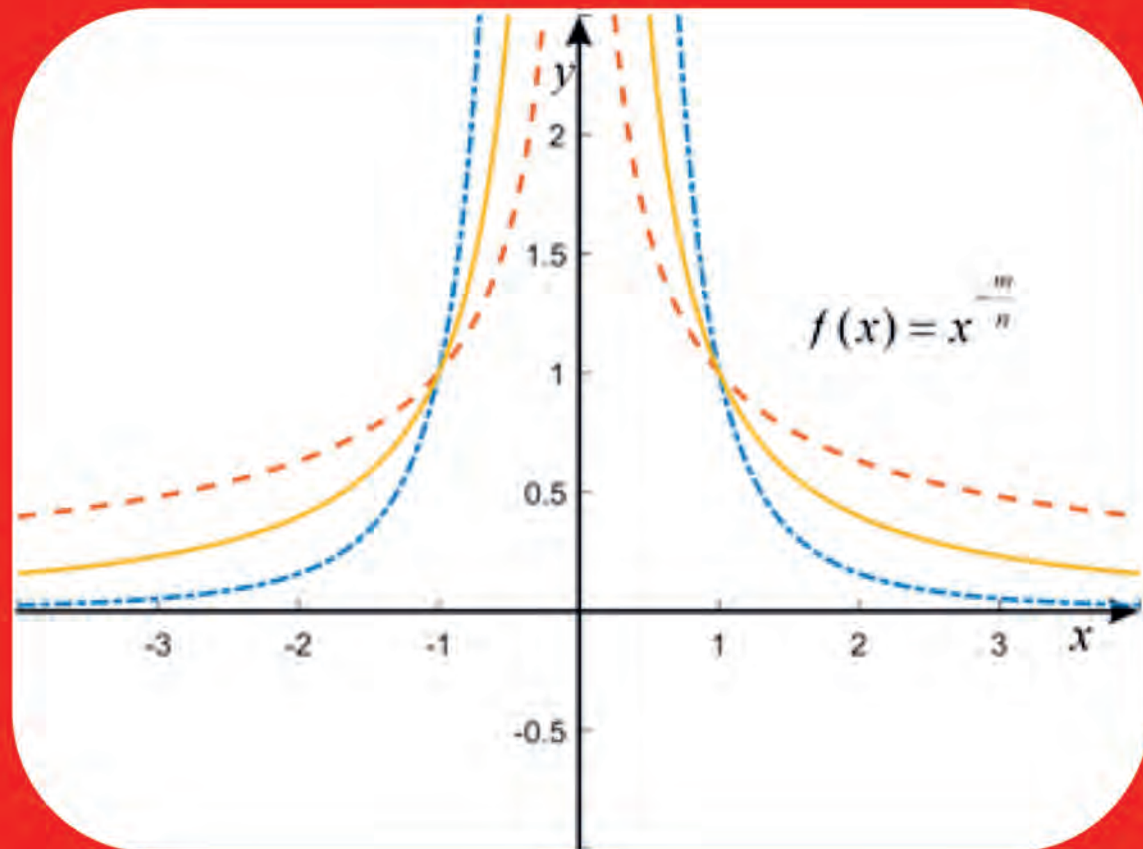

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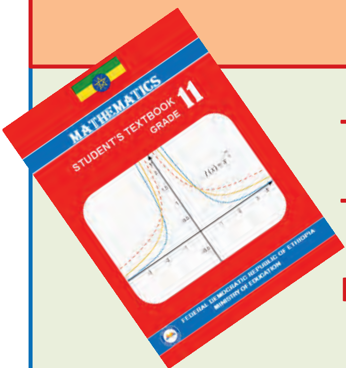

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MATHEMATICS

STUDENT'S TEXTBOOK GRADE 11

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Table of Contents

Welcoming Message to Grade 11 Students

Dear students, you are welcome to Grade 11 mathematics educations. This is a golden opportunity in your academic career. Joining this grade level is a new experience and a transition to Grade 12 mathematics educations. At this grade level, you are expected to receive new and advanced opportunities that can help you learn and grow in the field of Mathematics, in life and work. Enjoy it!

Introduction about the Students' Textbook

Dear students, this textbook is organized in such a way that there is an introductory remark about how to use the textbook, some suggestions about how to care the textbook so that it will serve for many beneficiaries for a long period of time, and basic units are addressed after the introduction section.

The Grade 11 textbook is organized having 8 units. The units include: Relations and Functions, Rational Expressions and Rational Functions, Matrices, Determinants and their Properties, Vectors, Transformation of the Plane, Statistics, and probability. Each unit is composed of an introduction, learning outcomes, key words, lessons, and summary and review exercise.

In a unit, a brief introduction about the unit is provided. Following a brief introduction of a unit, the expected learning outcomes are presented. Once the objectives are communicated, the different lessons of the unit are presented. In the end, summary and review exercise are packed, respectively.

As mentioned earlier, a unit in the textbook is divided into different lessons. In every lesson of each unit, you learn about mathematics frequently through five components. That is, structurally, the lessons in this textbook usually have five components; namely, Activity, Definitions/Theorems/Notes, Examples and Solutions, and Exercises.

The most important part in this process is to practice problems by yourself based on what your teacher shows and explains. Your teacher will also give you feedback and assistance, and facilitate further learning. In such a way, you will be able not only to acquire new knowledge and skills but also to develop them further. Herewith, a brief explanation of each sub-component of a lesson is forwarded.

Activity

This part of the lesson requires you to revise what you have learnt at different stages of your mathematics education and reflect on the topic under discussion by using this background knowledge. The activity also introduces you to what you are going to learn in the respective new lesson or topic.

Definition/Theorem/Note

This part of the textbook presents and explains new concepts/ definitions/theorems.

Example and Solution

Here, the textbook provides you specific examples and thereby helps you to improve your understanding of the new content. In this part, your teacher will give you explanations and/or solutions and you are, therefore, advised to listen to your teacher's explanations very carefully and participate actively in the process. You are also required to refer to the solution part of the textbook for your review and self-learning. In this sub-section, a student is further expected to study on the remaining examples that a classroom teacher will not explain during his/her classroom instruction.

Exercise

Under this part of the textbook, you will be required to solve the problems or questions that are given as exercise individually, in pairs or groups so as to further practice what you have learnt in the examples. That is, a teacher may provide selected exercise questions for you as class work, homework or project work. When you are doing the exercises either individually, in pairs or groups, you are expected to share your opinions with your friends, listen to others' ideas carefully and compare your ideas with others. However, it is always advisable to make some efforts individually before working in pair or groups in order to have a better understanding and input in the process.

In some of the cases, some exercises may not be covered by a classroom teacher. In that case, a student is expected to study the remaining items of the exercise a teacher will not cover during classroom instruction. A teacher will provide the possible answers for those items in the exercises after completing your study, and you are expected to check what you will do against the respective teacher's answers.

Checking in every step

Throughout a lesson, students shall check their progress and the respective teacher should do the same. It is also true that students are required to check the correct answers and solutions for the exercises at the end of each unit. If you want to practice more, therefore, you need to go to the review exercises. Doing the Review Exercises is always recommended since they help you develop a concrete view of the lesson contents.

Generally, students are advised to practice, drill and exercise each activity, example, and exercise on a daily basis as mastering mathematical skills take a long period of time and frequent practice to be second nature. You are also requested to read the notes with deep understanding.

UNIT 1

RELATIONS AND FUNCTIONS

Unit Outcomes

By the end of this unit, you will be able to:

- Generalize patterns using explicitly defined and recursively defined functions.
- Know the inverse of a given relation.
- Know types of functions.
- Recognize real valued functions.
- Know how to find compositions of functions.
- Recognize inverse of a function.
- Sketch the graph of the inverse function.
- Understand how to apply relation and function in real life situations.

Unit Contents

KEY Words

- **composition of functions**
- **domain**
- **function**
- **greatest integer (floor) function**
- **identity function**
- **inverse function**
- **modulus**
- **power function**
- **range**
- **relation**
- **signum (sgn) function**
- **vertical line test**

Introduction

In your daily life, you come across many patterns that characterize relationships between individuals such as brother and sister, father and daughter or son, teacher and student, etc. In mathematics also you come across many relationships such as a line l is perpendicular to line m; a number a is less than a number b; a number c is a factor of a number d, etc.

In this unit, the concepts about relations and functions that you have learned in Unit 1 of Grade 10 will be revised and some special types of functions such as power functions, absolute value functions, signum functions and the greatest integer functions and their properties will be discussed. Furthermore, some applications of relations and functions will also be considered.

1.1 Relations

Relation: Domain and Range

In this section, we will revise about the concept of a relation and determine domain and range of relations. Graphs of relations that are defined using numbers in the coordinate plane will also be considered.

Activity 1.1

1. Mention at least five different relationships between family members of a certain family.
2. Let $A = \{1, 2\}$ and $B = \{a, b\}$.
   a. Find all subsets of $A \times B$.
   b. Find the set of the first and the set of the second coordinates of all members of the sets in (a).

From your responses in Activity 1.1, observe that relationships between individuals form a set of ordered pairs of individuals and all the nonempty subsets of $A \times B$ are sets of ordered pairs.

Note

Any set of ordered pairs is a relation.

Definition 1.1

Given two sets A and B, any set R of ordered pairs (x, y), where $x \in A$ and $y \in B$, is called a **relation from A to B** and for $(x, y) \in R$, x is called the first component or coordinate and y is called the second component or coordinate of (x, y).

Given a relation R from A to B;

a. the set of all the first components or coordinates of the elements of R is called the **domain of R and it is denoted by Dom(R).**

That is, $\text{Dom}(R) = \{x \in A \mid (x, y) \in R \text{ for some } y \in B\}$.

b. the set of all the second components or coordinates of the elements of R is called the **range of R, denoted by Ran(R).**

That is, $\text{Ran}(R) = \{y \in B \mid (x, y) \in R \text{ for some } x \in A\}$.

Example 1

Find the domain and range of the relation R = {(45, 65), (48, 68), (41, 62), (46, 66), (50, 70)}.

Solution

Dom(R) = {41, 45, 46, 48, 50}, the set of all the first coordinates.

Ran(R) = {62, 65, 66, 68, 70}, the set of all the second coordinates.

Note

i. Given an ordered pair (x, y), x is called the first component or the first coordinate of the given ordered pair and y is called the second component or the second coordinate of the ordered pair (x, y).

ii. For an ordered pair (x, y), order is important; that is, $(x, y) = (y, x)$ only when $x = y$.

iii. For two ordered pairs (a,b) and (c,d); $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$.

Example 2

In a certain school, there are four sections of Grade 11 and each section is related with the number of students in the given section as in Table 1 below.

Table 1

Sections	Number of Students
Section 1	49
Section 2	51
Section 3	48
Section 4	50

a. Find the relation R defined by Table 1.

b. Find the domain and range of the relation defined by Table 1.

Solution

a. The information given in Table 1 can be represented as a set of ordered pairs. In this case, the first component represents the section and the second component represents the number of students in the given section. The ordered pairs (Section 1, 49), (Section 2, 51), (Section 3, 48) and (Section 4, 50) define a relation between a section and the number of students in the given section. This information is given as a set of ordered pairs by:

R={(Section 1, 49), (Section 2, 51), (Section 3, 48), (Section 4, 50)}.

b. Domain of R is given by Dom(R)={Section 1, Section 2, Section 3, Section 4}, which is the set of all the first components or coordinates of the elements of R and the range of R is given by Ran(R) = {49,51,48,50} , which is the set of all second components or coordinates of the elements of R.

Example 3

Given a relation $R = \{(x, y) : y = x^2 \text{ and } x \in \{-2, -1, 0, 1, 2\}\}$.

a. list all the elements of R;

b. find the domain and range of R.

Solution

a. When $x = -2, y = (-2)^2 = 4$, when $x = -1, y = (-1)^2 = 1$, when $x = 0, y = 0^2 = 0$, when $x = 1, y = 1^2 = 1$ and when $x = 2, y = 2^2 = 4$.

Thus, $R = \{(-2,4), (-1,1), (0,0), (1,1), (2,4)\}$.

b. Dom(R) = $\{-2, -1, 0, 1, 2\}$, the set of all the first coordinates of the elements of R and Ran{R}=$\{0,1,4\}$, the set of all the second coordinates of the elements of R.

Exercise 1.1

1. Let $R = \{(1,1), (2,-2), (10, 20), (5,25), (10,100)\}$. Find the domain and range of R.

2. Table 2 gives the marks of a certain student in four subjects out of 100.

Table 2

Subjects	Marks (out of 100)
English	89
Mathematics	91
Physics	88
Chemistry	90

a. List all the elements of a relation R defined by Table 2.

b. Find the domain and range of R in (a).

3. Let R be a relation defined by $R = \{(x, y) : y = x^2 - 2 \text{ and } x \in \{0, 1, 2, 3, 4, 5\}\}$.

a. list all the elements of R;

b. find the domain and range of R.

Representations of Relations (1)

A relation may consist of a finite set of ordered pairs or an infinite set of ordered pairs. Furthermore, a relation may be defined by several different methods, such as:

a. a set of ordered pairs;

b. a correspondence between domain and range;

c. a graph;

d. an equation;

e. an inequality or a combination of these , and so on.

Example 4

Given R= {(1, 2), (3, 4), (4, 4), (5, 6)}. Find the domain and range of R.

Solution

R is defined by a finite set of ordered pairs.

- Dom(R) = {1,3,4,5}, the set of all the first coordinates of the elements of R and
- Ran(R) = {2,4,6}, the set of all the second coordinates of the elements of R.

Example 5

A relation R is defined by a diagram in **Figure 1.1** below. Find the domain and range of R.

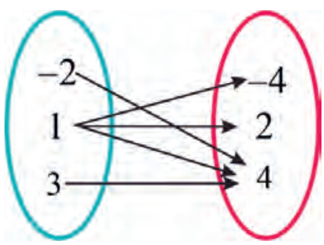


Figure 1.1

Solution

R, as a set of ordered pairs, is given by $R=\{(1,2),(1,4),(1,-4),(-2,4),(3,4)\}$.

Then

a. $\text{Dom}(R)=\{-2,1,3\}$, the set of all the first coordinates of elements of R;

b. $\text{Ran}(R)=\{-4,2,4\}$, the set of all the second coordinates of elements of R.

Example 6

Find the domain and range of the relation R defined by the graph given in **Figure 1.2**.

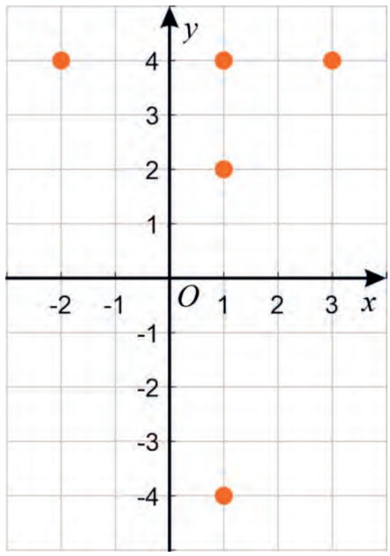


Figure 1.2

Solution

R as a set of ordered pairs is given by $R=\{(-2,4),(1,-4),(1,2),(1,4),(3,4)\}$. Then

a. $\text{Dom}(R)=\{-2,1,3\}$, the set of all the first coordinates of elements of R and

b. $\text{Ran}(R)=\{-4,2,4\}$, the set of all the second coordinates of elements of R.

Exercise 1.2

1. Find the domain and range of the relation

   $R = \{(-2,-5),(-1,-3),(0,-1),(1,1),(2,3)\}$.

2. List all the elements of the relation R defined by **Figure 1.3** and find its domain and range:

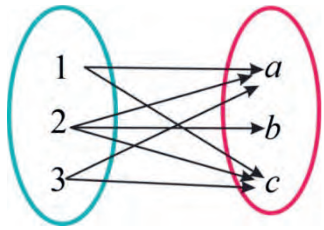


Figure 1.3

3. Find the domain and range of the relation R given in **Figure 1.4**.

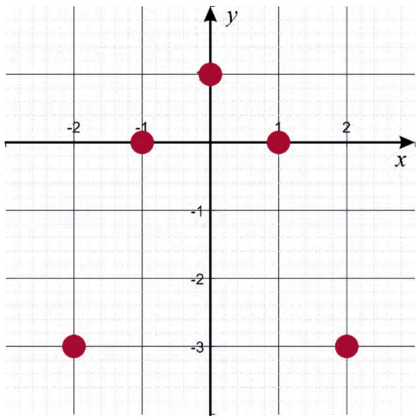


Figure 1.4

4. Find the domain and range of the relation, $R = \{(x, y): x, y \in \mathbb{R} \text{ and } y = x^2\}$.

Representations of Relations (2)

Example 7

A relation R is defined by $R = \{(x, y): x \in \mathbb{R}, y \in \mathbb{R} \text{ and } y = 2x\}$.

Find the domain and range of R.

Solution

R is defined as an infinite set of ordered pairs. From the definition of R, the first coordinate which is in the domain of R can be any real number and also the second coordinate which is in the range of R can be any real number, because for any $y \in \mathbb{R}$, you have $x = \frac{y}{2} \in \mathbb{R}$ and $(x, y) \in \mathrm{R}$.

Thus,

a. the domain of R is the set of all real numbers, $\mathbb{R}$ and

b. the range of R is the set of all real numbers, $\mathbb{R}$.

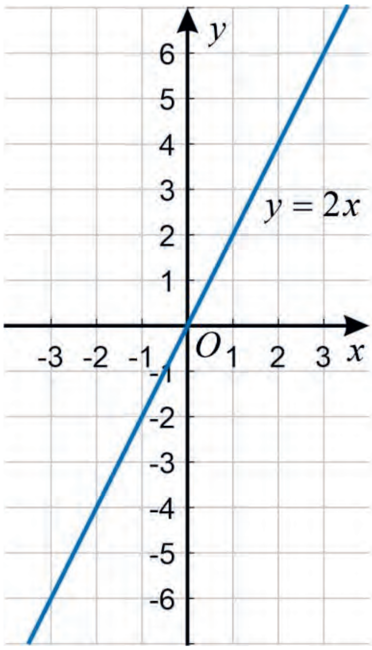


Figure 1.5

Example 8

Draw the graph of the relation R given by

$\mathrm{R} = \{(x, y) \mid x, y \in \mathbb{R}, y \leq x + 1 \text{ and } y + x \leq 2\}$. and find the domain and range of R.

Solution

First draw the lines $y = x + 1$ and $y + x = 2$ in the coordinated plane. Here, you can use a table of values. Then, the two lines divide the coordinate plane into four regions and by taking a point on each of the four regions, we can determine the region(s) that represents the graph of R as given in Figure 1.6 and from the graph;

a. $\text{Dom}(R) = \mathbb{R}$; b. $\text{Ran}(R) = \left(-\infty, \frac{3}{2}\right]$.

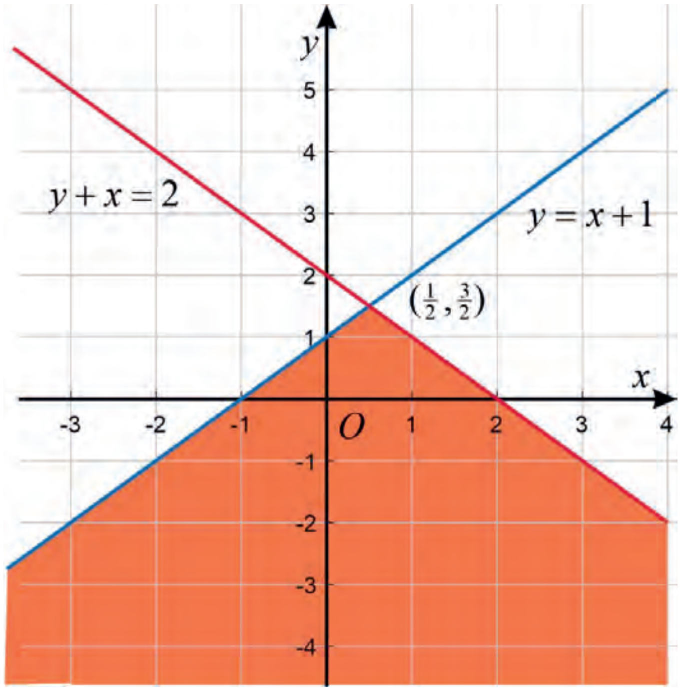


Figure 1.6

Exercise 1.3

1. Find domain and range of the relation $R = \{(x, y) : x, y \in \mathbb{R} \text{ and } y = x^2\}$.
2. Draw the graph of the relation R given by

   $R = \{(x, y) : x, y \in \mathbb{R}, y \leq x - 1 \text{ and } y + x \geq 3\}$ and find the domain and range of R.

1.2 Inverse of Relations and Their Graphs

Inverse of Relations

Activity 1.2

Consider the relations $R=\{(a,1),(b,2),(c,3),(d,4)\}$ and

$S=\{(1,a),(2,b),(3,c),(4,d)\}$.

a. Interchange the order of the first and the second components of the ordered pairs in R and obtain a new relation.

b. Interchange the order of the first and the second components of the ordered pairs in S and obtain a new relation.

From Activity 1.2, observe that from a given relation you can form another relation by interchanging the order of the first and the second components of the ordered pairs in the given relation and these two relations are called **inverse relations of each other.**

Definition 1.2

The inverse of a relation is a relation formed by interchanging the order components of each of the ordered pairs in the given relation.

That is, if $R=\{(x,y):(x,y)\in R\}$ is a relation, then the inverse of R, denoted by R^{-1}, is the relation defined by $R^{-1}=\{(y,x):(x,y)\in R\}$.

Example 1

Let $R=\{(1,3),(2,5)(3,7),(4,11),(5,23)\}$. Then, find R^{-1}.

Solution

The elements of R^{-1} are given by interchanging the order of the components of the elements of R. That is, $R^{-1}=\{(3,1),(5,2)(7,3),(11,4),(23,5)\}$.

Example 2

Table 3 below gives a relation R between a person's age and the person's maximum recommended heart rate. Find R^{-1}.

Table 3

Age (years) *x*	**Maximum Recommended Heart Rate (Beats per Minute) *y***
20	200
30	190
40	180
50	170
60	160

Solution

The relation R is given as a set of ordered pairs by:

$$R = \{(20,200),(30,190),(40,180),(50,170),(60,160)\}.$$

Then, the inverse of R is given by interchanging the order of the first and the second coordinates of ordered pairs of R; that is,

$$R^{-1} = \{(200,20),(190,30),(180,40),(170,50),(160,60)\}.$$

Example 3

For the relation R defined by the diagram in **Figure 1.7**, find R^{-1}.

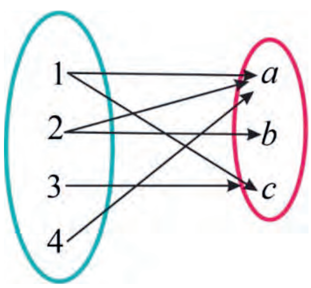


Figure 1.7

Solution

The relation R is defined by a diagram and from the diagram in **Figure 1.7** R is given by $R = \{(1, a), (1, c), (2, a), (2, b), (3, c), (4, a)\}$.

Then, $R^{-1} = \{(a,1), (c,1), (a,2), (b,2), (c,3), (a,4)\}$. It is obtained by interchanging the order of the first and the second components of all the ordered pairs in R.

Example 4

Let $R = \{(x, y) : x, y \in \mathbb{R} \text{ and } y = 3x - 5\}$.. Find R^{-1}.

Solution

$$R^{-1} = \{(x, y) : x, y \in \mathbb{R} \text{ and } x = 3y - 5\} = \{(x, y) : x, y \in \mathbb{R} \text{ and } y = \frac{x+5}{3}\}.$$

Exercise 1.4

Find the inverse of each of the following relations.

1. $R = \{(1,1), (2,4), (3,9), (4,16), (5,25)\}$.
2. R is defined by the diagram given in Figure 1.8.

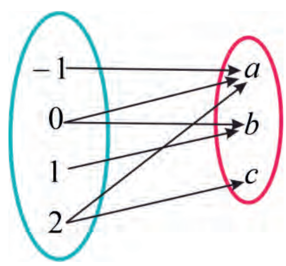


Figure 1.8

3. $R = \{(x, y) : x, y \in \mathbb{R} \text{and } y = 4x + 2\}$

Domain and Range of Inverse Relations

Activity 1.3

Consider $R = \{(2,20),(3,19),(4,18),(5,17),(6,16)\}$.

a. Find R^{-1}

b. Find the domain and range of R.

c. Find the domain and range of R^{-1}

From Activity 1.3, observe the following results for the given relation R.

a. Domain of R^{-1} = Range of R;

b. Range of R^{-1} = Domain of R.

As relations are often specified by expressions involving first component, say x, and a second component, say y, it is natural for us that we want to work with the concept of the inverse relation in that setting. The inverse relation can be formed by interchanging the roles of x and y in the defining expression.

That is, given a relation $R = \{(x,y) \mid (x,y) \in R\}$, we have two ways of defining R^{-1};

a. interchanging x and y in the ordered pair (x, y) and keep the relation; that is,

$R^{-1} = \{(y,x) : (x,y) \in R\}$.

b. keep the ordered pair (x, y) as it is and interchange the roles of x and y in the relation; that is, $R^{-1} = \{(x,y) : (y,x) \in R\}$.

Example 5

Let $R = \{(-3,2),(-2,3),(-1,4),(0,5),(1,6),(2,7),(3,8)\}$. Then find the domain and the range of R^{-1}.

Solution

R^{-1} is obtained by interchanging the order of the components of each of the ordered points in R.

That is, $R^{-1} = \{(2,-3),(3,-2),(4,-1),(5,0),(6,1),(7,2),(8,3)\}$ and then,

i. the domain of R^{-1} is $\text{Dom}(R^{-1}) = \{2,3,4,5,6,7,8\}$ and

ii. the range of R^{-1} is $\text{Ran}(R^{-1}) = \{-3,-2,-1,0,1,2,3\}$.

Example 6

Given $R = \{(x,y) \mid x, y \in \mathbb{R} \text{ and } y = 3x+6\}$.

a. Find the inverse of R.

b. Find the domain and range of R^{-1}.

Solution

a. First interchange variables x and y in the equation $y = 3x+6$ to obtain $x = 3y+6$. Then solving for y in terms of x gives you $y = \frac{1}{3}x - 2$.

Thus, $R^{-1} = \left\{(x,y) \mid x, y \in \mathbb{R} \text{ and } y = \frac{1}{3}x - 2\right\}$.

R^{-1} can also be written as $R^{-1} = \{(y,x) \mid x, y \in \mathbb{R} \text{ and } y = 3x+6\}$.

Observe that the two expressions of R^{-1} define the same relation.

b. $\text{Dom}(R) = \mathbb{R} = \text{Ran}(R^{-1})$ and $\text{Ran}(R) = \mathbb{R} = \text{Dom}(R^{-1})$.

Example 7

Find the inverse of $R = \{(x,y) \mid x, y \in \mathbb{R}, y \geq x-3 \text{ and } y \leq x+5\}$.

Solution

$R^{-1} = \{(x, y) \mid x, y \in \mathbb{R},\ x \geq y - 3 \text{ and } x \leq y + 5\}$

$= \{(x, y) \mid x, y \in \mathbb{R}, -y \geq -x - 3 \text{ and } -y \leq -x + 5\}$ (Solving for y in terms of x).

$= \{(x, y) \mid x, y \in \mathbb{R},\ y \leq x + 3 \text{ and } y \geq x - 5\}$.

That is, $R^{-1} = \{(x, y) \mid x, y \in \mathbb{R},\ y \leq x + 3 \text{ and } y \geq x - 5\}$

Exercise 1.5

Find the inverse and also the domain and range of the inverse of each of the following relations.

a. $R = \{(1,3), (2,4), (3,9)\}$

b. $S = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } y = 2x - 4\}$

c. $T = \{(x, y) \mid x, y \in \mathbb{R}, y \geq 2x + 1 \text{ and } y \leq x - 2\}$

Graphs of Inverse Relations

Activity 1.4

Consider the ordered pairs $(1,2), (-3,1), (-2,-4)$ and $(2,-3)$.

1. Plot the given points on the coordinate plane;
2. Find the image of each of the ordered pairs given with reflection of the coordinate plane on the line $y = x$.

From your responses in Activity 1.4, observe that the image of any given ordered pair (a, b) with respect to the reflection of the coordinate plane on the line $y = x$ is (b, a).

The graphs of relations and their inverses are related in an interesting way. First, note that, in the coordinate system ordered pairs (a, b) and (b, a) are the images of each other with respect to the reflection of the coordinate plane on the line $y = x$.

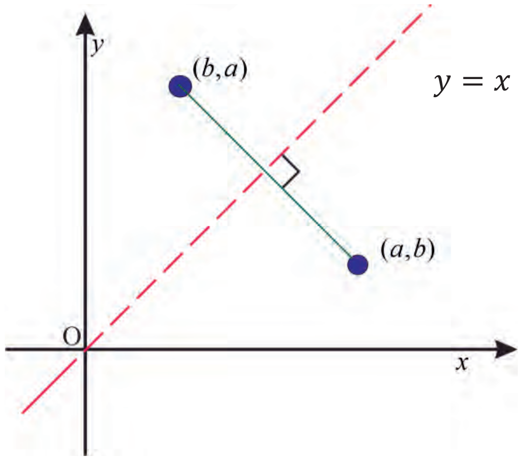


Figure 1.9: Image of (a,b) with respect to the reflection on the line $y = x$.

Therefore, the graph of the inverse of a given relation is the image of the graph of R with respect to the reflection of the coordinate plane on the line $y = x$.

Example 8

Given $R = \{(0,0),(1,2),(2,4),(3,6)\}$, plot the graphs of R and R^{-1} on the same coordinate plane.

Solution

First let us find R^{-1} by interchanging the order of the components of the ordered pairs of R and obtain $R^{-1} = \{(0,0),(2,1),(4,2),(6,3)\}$.

Then, plotting each of the ordered pairs in both R and R^{-1} gives us the following figure.

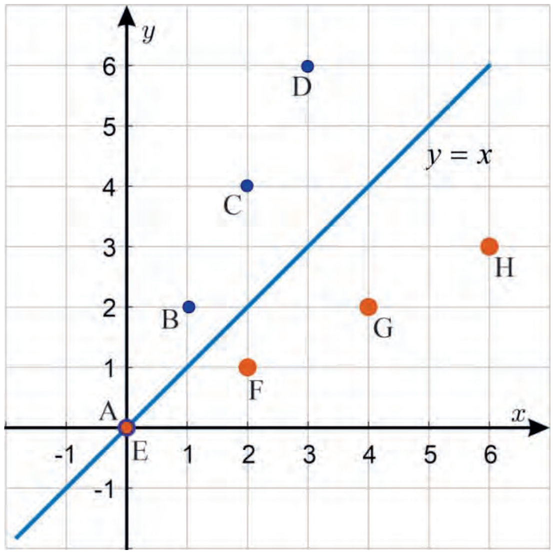


Figure 1.10: Graphs of R and R^{-1}

In Figure 1.10, the points E, F G and H are images of the points A, B, C and D after they are reflected with respect to the line $y = x$.

Example 9

Draw the graph of the inverse of the relation, $R = \{(x, y) | x, y \in \mathbb{R} \text{ and } y = 3x + 6\}$ in the coordinate plane.

Solution

First $R^{-1} = \{(x, y) | x, y \in \mathbb{R} \text{ and } x = 3y + 6\}$ and solving for y in terms of x gives you

$$R^{-1} = \left\{(x, y) | x, y \in \mathbb{R} \text{ and } y = \frac{1}{3}x - 2\right\}.$$

First, draw the graph of R and reflect it with respect to the line $y = x$ and obtain the graph of R^{-1} as in **Figure 1.11.**

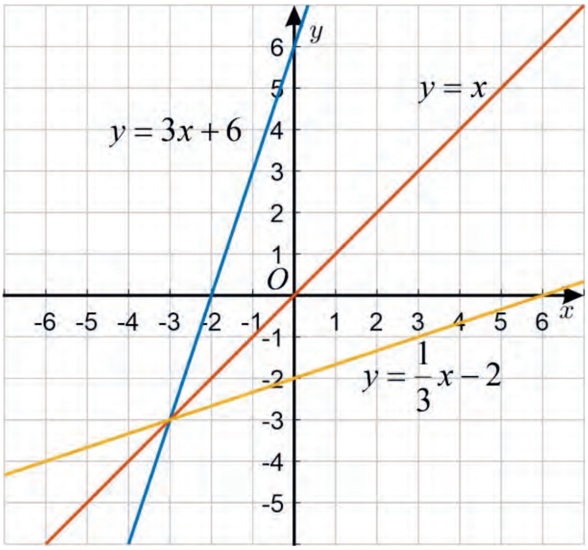


Figure 1.11: Graphs of $y = 3x + 6$ and $y = \frac{1}{3}x - 2$.

Example 10

Graph $R = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } y = x - 4\}$ and its inverse in the same coordinate plane.

Solution

First, let us find R^{-1}.

Thus, $R^{-1} = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } x = y - 4\} = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } y = x + 4\}$. Then, we draw the graph of R in the coordinated plane and reflect it with respect to the line $y = x$ to obtain the graph of R^{-1} as in **Figure 1.12**.

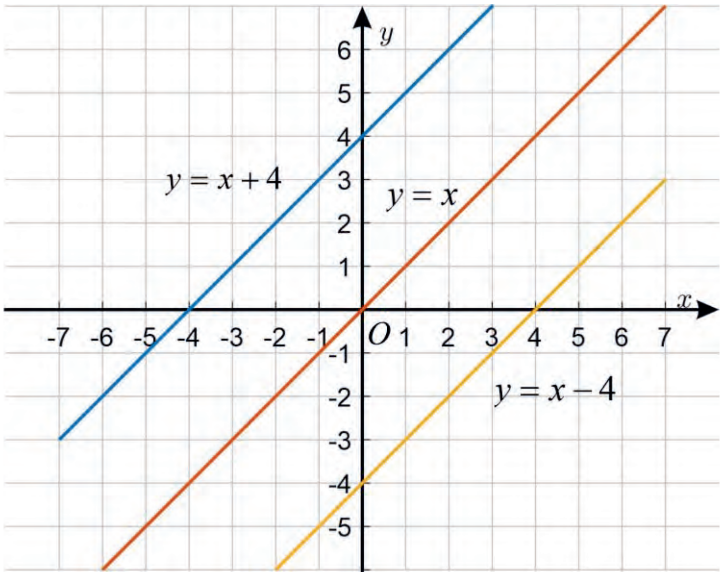


Figure 1.12: Graphs of $y = x + 4$ and $y = x - 4$,

Exercise 1.6

Draw the graph of the given relation and its inverse on the same coordinate plane for each of the following relations.

a. $R = \{(2,4), (3,9), (4,16)\}$.

b. $T = \{(x, y) | x, y \in \mathbb{R} \text{ and } y = x + 5\}$.

c. $S = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } y = 2x - 2\}$.

1.3 Types of Functions

In this section, you will be introduced with special types of relations called functions. Some special types of functions: power functions, absolute value functions, signum functions and the greatest integer function will be considered.

Functions

Activity 1.5

What differences do you observe between the relations R_1 and R_2 given in **Figure 1.13**?

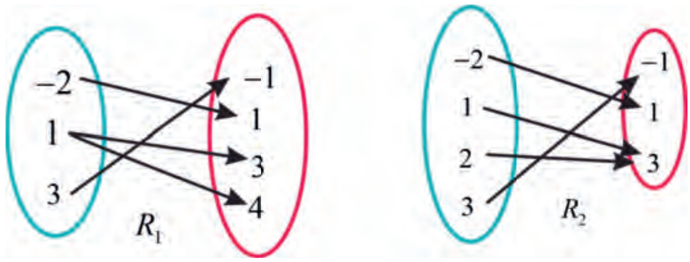


Figure 1.13

From your responses in Activity 1.5, observe that:

i. there are relations in which one first component is related to two or more different second components.

ii. there are relations in which every first component is related to only one corresponding second coordinate.

Relations of the second type are called functions of which its formal definition is given as follows.

Definition 1.3

A relation in which each object from the set of the first components of ordered pairs of the relation is related with exactly one object from the set of second components of the ordered pairs of the relation is called a **function**.

Example 1

Determine if $R=\{(-1,0),(0,-3),(2,-3),(3,0),(4,5)\}$ is a function.

Solution

From the ordered pairs in R, you can see that every first component is related to exactly one second component; that is, −1 is related with 0 only, 0 is related with −3 only, 2 is related with only −3, 3 is related with 0 only and 4 is related with only 5.

Therefore, R is a function.

Example 2

Determine if $R=\{(6,1),(7,3),(0,2),(6,4)\}$ is a function.

Solution

$\text{Dom}(R)=\{6,7,0\}$ and $\text{Ran}(R)=\{1,2,3,4\}$.

From the set of first components, you can see that, the relation has two ordered pairs with 6 as a first component: (6,1) and (6,4), but $1\neq 4$.

Therefore, the relation R is not a function.

Example 3

Show that the relation $R=\{(x,y): y=x+1\}$ is a function.

Solution

For any given variable x, there is only one variable $y = x + 1$. That is, if (x,y) and (x,z) are in R, then $y=x+1$ and $z=x+1$. This implies, $y = z$.

Example 4

Let $R = \{(x, y) : x, y \in \mathbb{R} \text{ and } x^2 - y^2 = 0\}$. Show that R is not a function.

Solution

$(1,-1)$ and $(1,1)$ are both in R, because $1^2 - (-1)^2 = 1 - 1 = 0$ and $1^2 - 1^2 = 1 - 1 = 0$, but $-1 \neq 1$. Thus, R is not a function.

Exercise 1.7

Which of the following relations are functions?

a. $R_1 = \{(1,1),(2,3),(3,4),(5,5)\}$

b. $R_2 = \{(1,0),(1,3),(4,5),(6,6)\}$

c. $R_3 = \{(x, y) : x, y \in \mathbb{R} \text{ and } y = x + 3\}$

d. $R_4 = \{(x, y) : x, y \in \mathbb{R} \text{ and } x^2 + y^2 = 1\}$

Vertical Line Test

Notation

If f is a function from A to B, then we denote it by $f : A \to B$ or $A \xrightarrow{f} B$ and

a. Dom(f) = A

b. Range of f is a subset of B.

c. If $(x, y) \in f$, then we write $y = f(x)$ and $f(x)$ is read as f of x.

Note

For any given relation R, to determine whether it is a function or not, check whether the relation gives two different second coordinates for the same first coordinate. In other words, given a relation R, if $(x, y) \in R$ and $(x, z) \in R$ implies $y = z$ for any x, y and z, then R is a function. Otherwise, R is not a function.

Example 5

Let $R = \{(x, y): x, y \in \mathbb{R} \text{ and } y = 2x+1\}$. Then, show that R is a function.

Solution

Suppose $(x, y) \in R$ and $(x, z) \in R$. Then $y = 2x+1$ and $z = 2x+1$.

This implies $y = 2x+1 = z$ and hence R is a function.

Note

The Vertical Line Test

A set of points in the coordinate plane represents a function if and only if no two points of the given set are on the same vertical line.

Example 6

Use the Vertical Line Test to determine if each one of the relations in the following graphs is function.

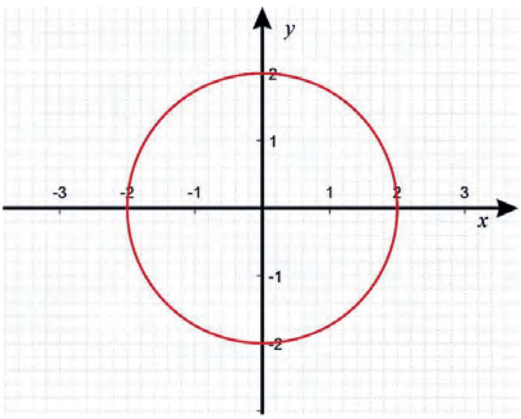


Figure 1.14: Graph of a relation R

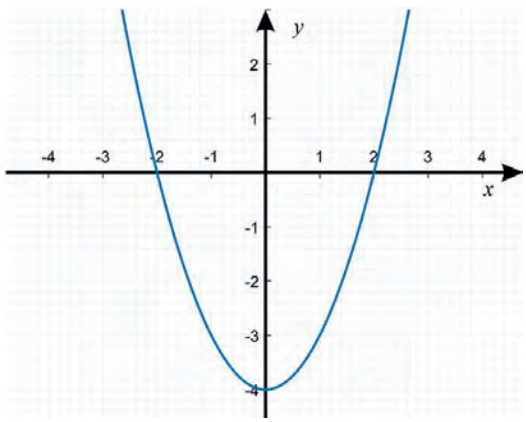


Figure 1.15: Graph of a relation S

Solution

a. You can see from the graph of R that it is possible to find a vertical line intersecting the graph more than once as it can be seen in **Figure 1.16**.

Hence, R does not represent a function.

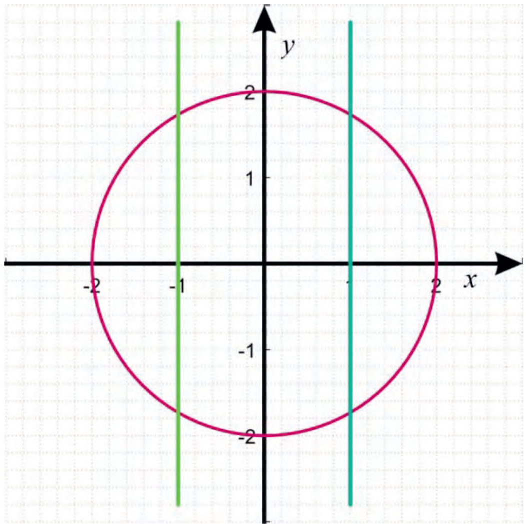


Figure 1.16

b. Any vertical line intersects the graph of S exactly once as shown in **Figure 1.17**. Hence, S is a function.

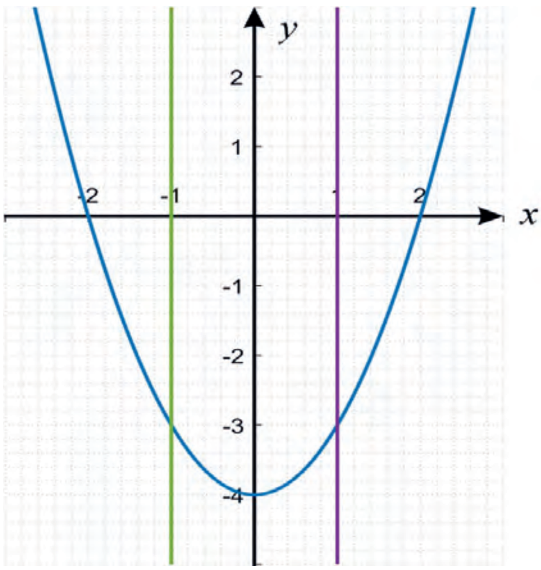


Figure 1.17

Exercise 1.8

1. Determine if the following relations are functions.
   a. $R_1 = \{(x, y) | x, y \in \mathbb{R} \text{ and } y = 3x + 1\}$
   b. $R_2 = \{(x, y): x, y \in \mathbb{R} \text{ and } x^2 + y^2 = 9\}$
2. Which of the following graphs represent graph(s) of a function?

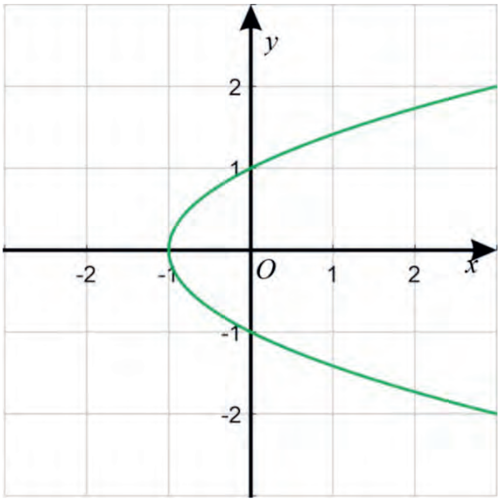


Figure 1.18

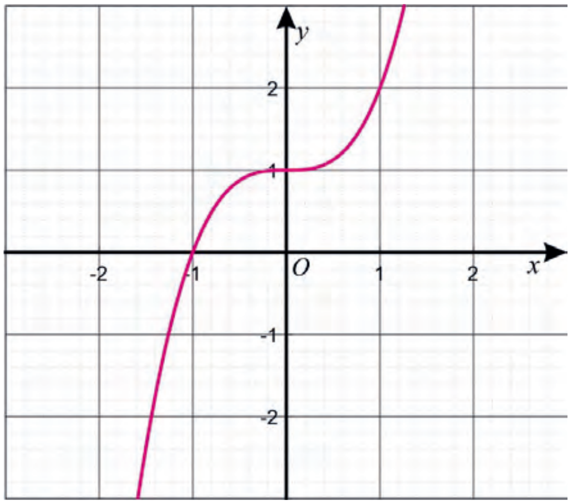


Figure 1.19

1.3.1 Power Functions with Their Graphs

Basic Properties of Power Functions (1)

Activity 1.6

a. A square is cut out of a cardboard, with each side having length x. Find the area of the square in terms of its side length x.
b. Find the volume of a cube with each edge having length t.
c. Find the area of a circle of radius r.

From your responses in Activity 1.6, observe that:

a. the area of the square is given as a function of x, which is a power of x;
b. the volume of the cube is given as a function of t, which is a power of t;

c. the area of the circle is given as a function of r, which is a constant multiple of power of r.

These three functions are examples of power functions; that means, they are functions that are some constant times a power of the given variable.

Definition 1.4

A function of the form $f(x) = ax^r$ for some nonzero real number a and a real number r is called a power function.

Example 1

Which of the following functions are power functions?

a. $f(x) = 3x^2$ b. $g(x) = 5x^{\frac{2}{3}}$ c. $h(x) = 2^x$

d. $l(x) = x^{\frac{4}{3}}$ e. $m(x) = \log_3 x$ f. $n(x) = -4x^{-3}$

Solution

The functions $f(x)$, $g(x)$, $l(x)$ and $n(x)$ are power functions whereas $h(x)$ and $m(x)$ are not a power functions; $h(x)$ is an exponential function and $m(x)$ is a logarithmic function.

Next, you will learn the behaviors of power functions and these will be done by considering different examples of power functions and generalize the behavior of these power functions.

Example 2

Find the domain and range of each of the following functions and draw their graphs.

a. $f(x) = x^2$ b. $g(x) = x^4$

Solution

a. Dom(f) = $\mathbb{R}$ and Ran(f) = $[0, \infty)$.

b. Dom(g)= $\mathbb{R}$ and Ran(g) = $[0, \infty)$.

To draw the graphs of both f and g, use the table of values given below.

x	-2	-1	0	1	2
$f(x)$	4	1	0	1	4
$g(x)$	16	1	0	1	16

Then, using the values in the table, we draw the graphs of f and g as in **Figure 1.20**.

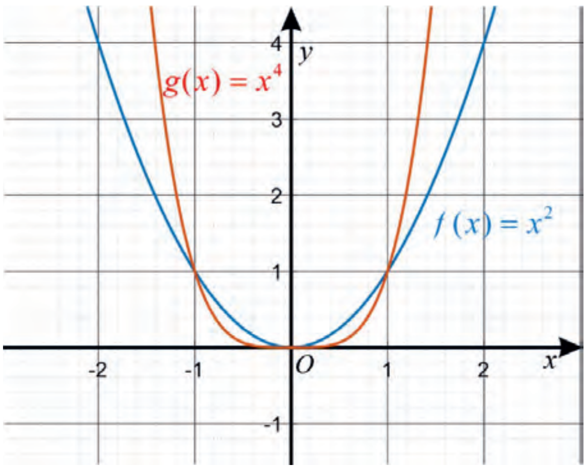


Figure 1.20: Graphs of $f(x) = x^2$ and $g(x) = x^4$

Using f and g in Example 2, we can generalize the properties of power functions of the form $f(x) = x^n$ for an even natural number n as follows.

I. Functions of the form $f(x) = x^n$ for an even natural number n have similar properties and some of these common properties are given as follows:

a. Domain of the function is $\mathbb{R}$; b. Range of the function is $[0, \infty)$;

b. The graph of the function is a curve that is similar to the curves in **Figure 1.21**.

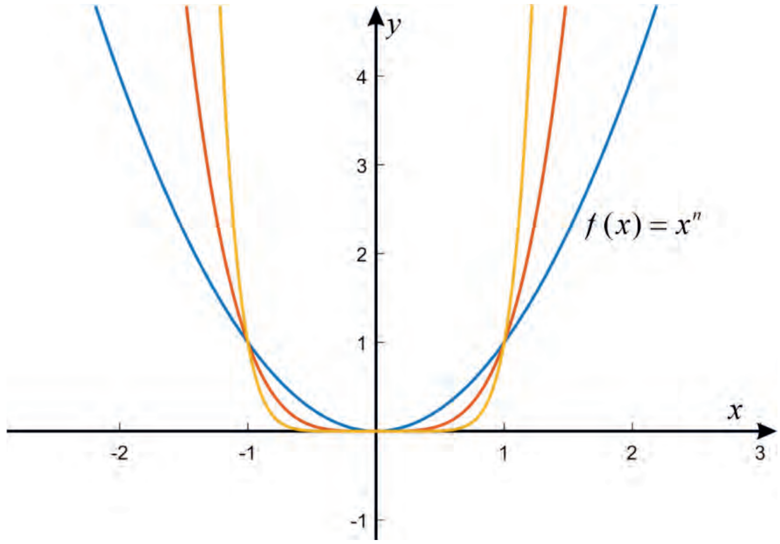


Figure 1.21: Graph of $f(x)=x^n$ for a positive even integer n.

Example 3

Find the domain and range of each of the following functions and draw their graphs.

a. $f(x)=x^3$ b. $g(x)=x^5$

Solution

a. $\text{Dom}(f)=\mathbb{R}$ and $\text{Ran}(f)=\mathbb{R}$.

b. $\text{Dom}(g)=\mathbb{R}$ and $\text{Ran}(g)=\mathbb{R}$.

To draw the graphs of both f and g, use the table of values given below.

x	-2	-1	0	1	2
$f(x)$	-8	-1	0	1	8
$g(x)$	-32	-1	0	1	32

Then, using the values in the table, we draw the graphs of f and g as in **Figure 1.22.**

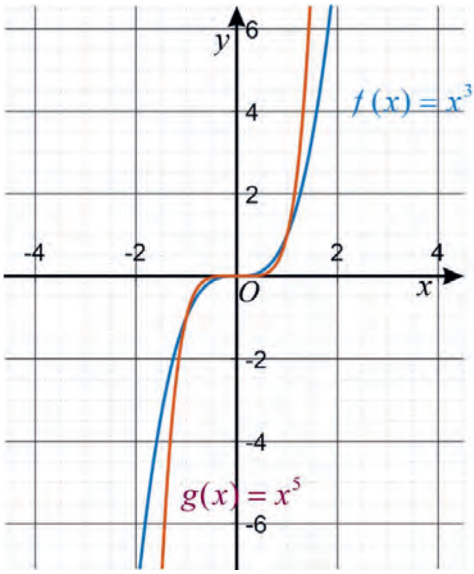


Figure 1.22: Graphs of $f(x) = x^3$ and $g(x) = x^5$

II. Functions of the form $f(x) = x^n$ for an odd natural number n have similar properties and some of these common properties are given as follows:

a. Domain of the function is $\mathbb{R}$;

b. Range of the function is $\mathbb{R}$;

c. The graph of the function is a curve that has the same shape as the curves in **Figure 1.23.**

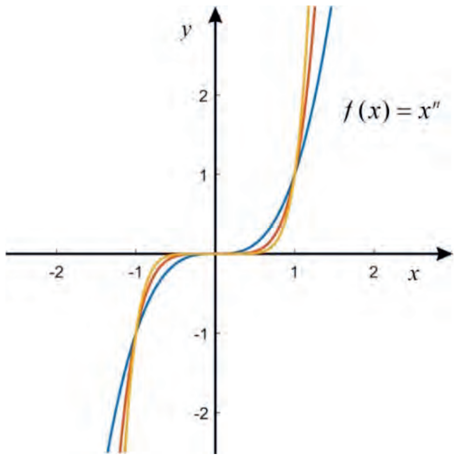


Figure 1.23: Graph of $f(x) = x^n$ for a positive odd integer n.

Exercise 1.9

For each of the following functions, find the domain, range and sketch the graph.

a. $f(x) = x^6$ b. $g(x) = x^8$ c. $h(x) = x^7$ d. $l(x) = x^9$

Basic Properties of Power Functions (2)

Example 4

Find the domain and range of each of the following functions and draw their graphs.

a. $f(x) = x^{\frac{1}{2}}$　　　　b. $g(x) = x^{\frac{1}{4}}$

Solution

a. $\text{Dom}(f) = [0, \infty)$ and $\text{Ran}(f) = [0, \infty)$.

b. $\text{Dom}(g) = [0, \infty)$ and $\text{Ran}(g) = [0, \infty)$.

To draw the graphs of both functions *f* and *g*, use the table given below.

x	0	1	$\cdots$	4	$\cdots$	16
$f(x)$	0	1	$\cdots$	2	$\cdots$	4
$g(x)$	0	1	$\cdots$	$\sqrt[4]{4}$	$\cdots$	2

Then, using the values in the table, the graphs of f and g are given in **Figure 1.24**.

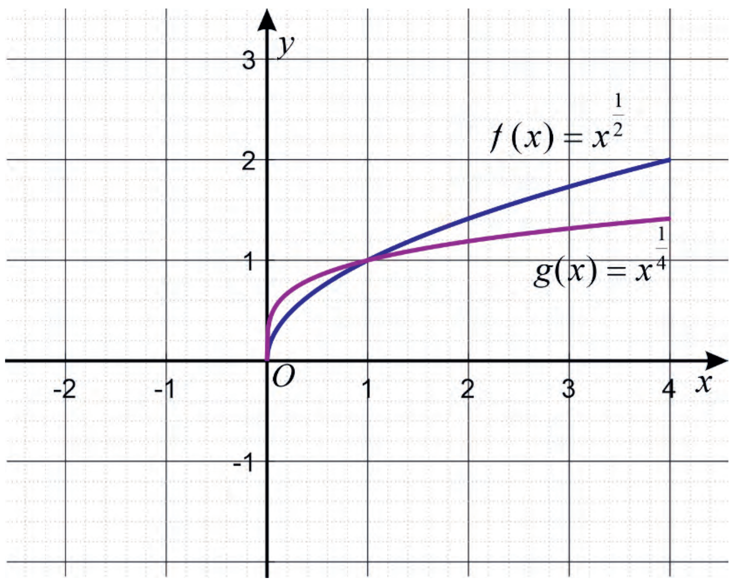


Figure 1.24: Graphs of $f(x) = x^{\frac{1}{2}}$ and $g(x) = x^{\frac{1}{4}}$

III. Functions of the form $f(x) = x^{\frac{1}{n}}$ for an even natural number n have similar properties and some of these common properties are:

a. Domain of the function is $[0, \infty)$;

b. Range of the function is $[0, \infty)$;

c. The graph of the function is given in **Figure 1.25**.

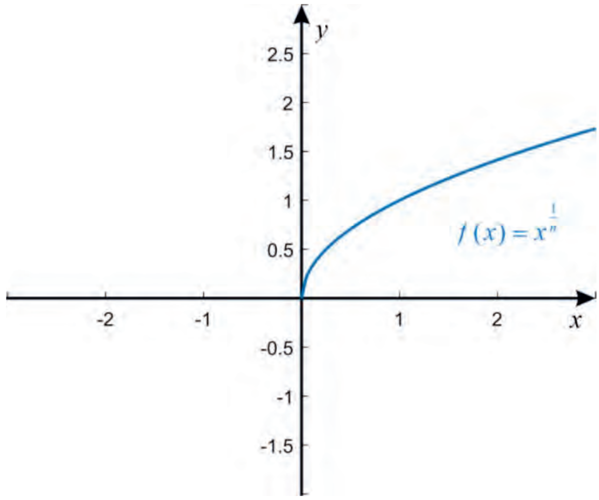


Figure 1.25: Graph of $f(x) = x^{\frac{1}{n}}$, where n is an even natural number.

Example 5

Find the domain and range of each of the following functions and draw their graphs.

a. $f(x) = x^{\frac{1}{3}}$

b. $g(x) = x^{\frac{1}{5}}$

Solution

a. $\text{Dom}(f) = \mathbb{R}$ and $\text{Ran}(f) = \mathbb{R}$.

b. $\text{Dom}(g) = \mathbb{R}$ and $\text{Ran}(g) = \mathbb{R}$.

To draw the graphs of both functions f and g, use the tables given below.

x	-32	$\cdots$	-8	$\cdots$	-1	0	1	$\cdots$	8	$\cdots$	32
$f(x)$	$\sqrt[3]{-32}$	$\cdots$	-2	$\cdots$	-1	0	1	$\cdots$	2	$\cdots$	$\sqrt[3]{32}$
$g(x)$	-2	$\cdots$	$\sqrt[5]{-8}$	$\cdots$	-1	0	1	$\cdots$	$\sqrt[5]{8}$	$\cdots$	2

Then, using the values in the table, the graphs of f and g are given in **Figure 1.26.**

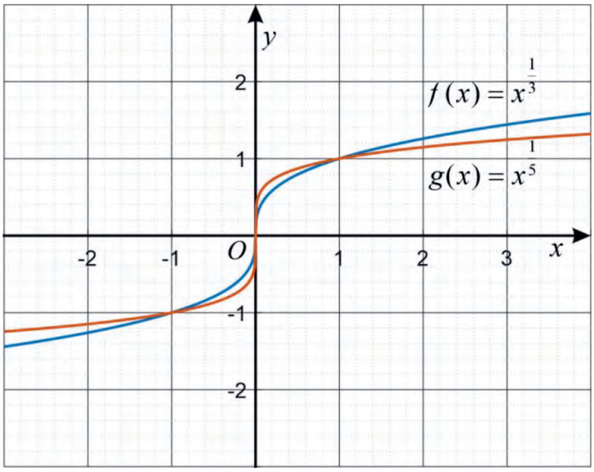


Figure 1.26: Graphs of $f(x)=x^{\frac{1}{3}}$ and $g(x)=x^{\frac{1}{5}}$

IV. Functions of the form $f(x)=x^{\frac{1}{n}}$ for an odd natural number n have similar properties and some of these common properties are:

a. Domain of the function is $\mathbb{R}$;

b. Range of the function is $\mathbb{R}$;

c. The graph of the function is a curve that has the same shape as the curve in **Figure 1.27**.

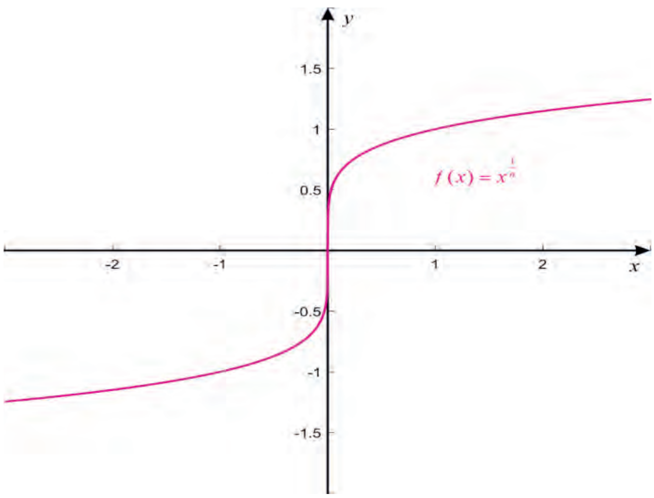


Figure 1.27: Graph of $f(x)=x^{\frac{1}{n}}$, where n is an odd natural number.

Exercise 1.10

For each of the following functions, find the domain, range and sketch the graph.

a. $f(x) = x^{\frac{1}{6}}$ b. $g(x) = x^{\frac{1}{8}}$ c. $h(x) = x^{\frac{1}{7}}$ d. $l(x) = x^{\frac{1}{9}}$

Basic Properties of Power Functions (3)

Example 6

Find the domain and range of each of the following functions and sketch their graphs.

a. $f(x) = x^{\frac{3}{2}}$ b. $g(x) = x^{\frac{5}{4}}$

Solution

a. $\text{Dom}(f) = [0, \infty)$ and $\text{Ran}(f) = [0, \infty)$.

b. $\text{Dom}(g) = [0, \infty)$ and $\text{Ran}(g) = [0, \infty)$.

To draw the graphs of both functions f and g, use the table given below,

x	0	1	⋯	4	⋯	16
$f(x)$	0	1	⋯	8	⋯	64
$g(x)$	0	1	⋯	$4\sqrt{2}$	⋯	32

Then, using the values in the table, the graphs of f and g are given in **Figure 1.28.**

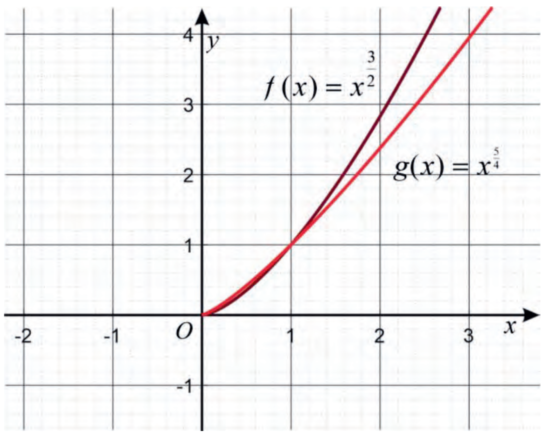


Figure 1.28: Graphs of $f(x) = x^{\frac{3}{2}}$ and $g(x) = x^{\frac{5}{4}}$

V. Functions of the form $f(x) = x^{\frac{m}{n}}$ for an odd natural number m and an even natural number n have similar properties and some of these common properties are:

a. Domain of the function is $[0, \infty)$.

b. Range of the function is $[0, \infty)$.

c. The graph of the function is a curve that has the same shape as the curves in **Figure 1.29.**

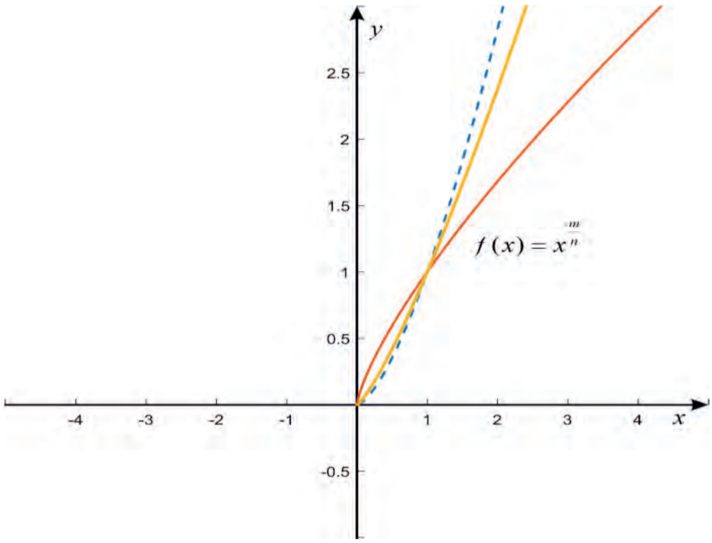


Figure 1.29: Graph of $f(x) = x^{\frac{m}{n}}$, where m and n are odd and even natural numbers respectively.

Example 7

Find the domain and range of each of the following functions and sketch their graphs.

a. $f(x) = x^{\frac{2}{3}}$

b. $g(x) = x^{\frac{4}{5}}$

Solution

a. $\text{Dom}(f) = \mathbb{R}$ and $\text{Ran}(f) = [0, \infty)$.

b. $\text{Dom}(g) = \mathbb{R}$ and $\text{Ran}(g) = [0, \infty)$.

To draw the graphs of both functions f and g, use the table given below,

x	-32	$\cdots$	-8	$\cdots$	-1	0	1	$\cdots$	8	$\cdots$	32
$f(x)$	$4\sqrt{4}$	$\cdots$	4	$\cdots$	1	0	1	$\cdots$	4	$\cdots$	$4\sqrt{4}$
$g(x)$	16	$\cdots$	$\sqrt[5]{8^4}$	$\cdots$	1	0	1	$\cdots$	$\sqrt[5]{8^4}$	$\cdots$	16

Then, using the values in the table, the graphs of f and g are given in **Figure 1.30.**

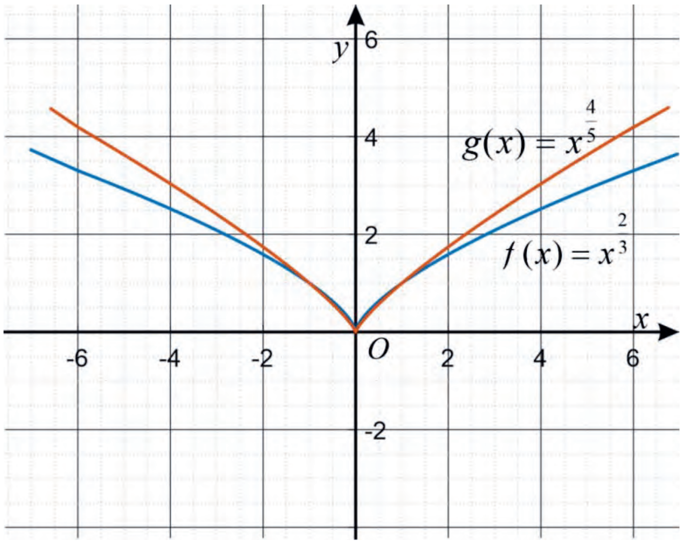


Figure 1.30: Graphs of $f(x) = x^{\frac{2}{3}}$ and $g(x) = x^{\frac{4}{5}}$

VI. Functions of the form $f(x) = x^{\frac{m}{n}}$ for an even natural number m and an odd natural number n have similar properties and some of these common properties are:

a. Domain of the function is $\mathbb{R}$.

b. Range of the function is $[0, \infty)$.

c. The graph of the function is a curve that has the same shape as the curves in **Figure 1.31.**

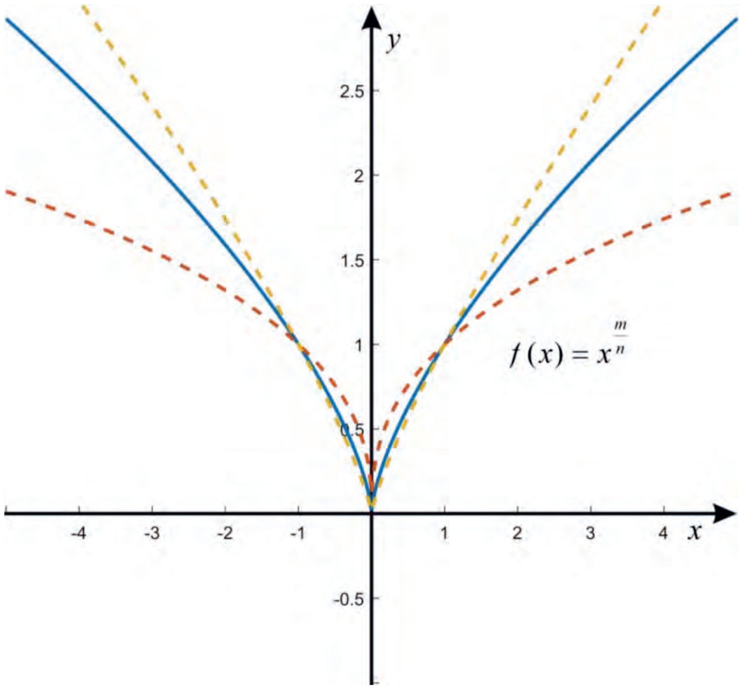


Figure 1.31: Graph of $f(x) = x^{\frac{m}{n}}$ for an even natural number m and an odd natural number n

Exercise 1.11

For each of the following functions, find the domain, range and sketch the graph.

a. $f(x) = x^{\frac{5}{6}}$

b. $g(x) = x^{\frac{7}{8}}$

c. $h(x) = x^{\frac{6}{5}}$

d. $l(x) = x^{\frac{4}{7}}$

Basic Properties of Power Functions (4)

Example 8

Find the domain and range of each of the following functions and sketch their graphs.

a. $f(x) = x^{-\frac{1}{2}}$

b. $g(x) = x^{-\frac{1}{4}}$

Solution

a. $\text{Dom}(f) = (0, \infty)$ and $\text{Ran}(f) = (0, \infty)$.

b. $\text{Dom}(g) = (0, \infty)$ and $\text{Ran}(g) = (0, \infty)$.

To draw the graphs of both functions f and g, use the table given below.

x	1	⋯	4	⋯	16
$f(x)$	1	⋯	0.5	...	0.25
$g(x)$	1	⋯	$\frac{1}{\sqrt{2}}$	⋯	0.5

Then, using the values in the table, the graphs of f and g are given in **Figure 1.32.**

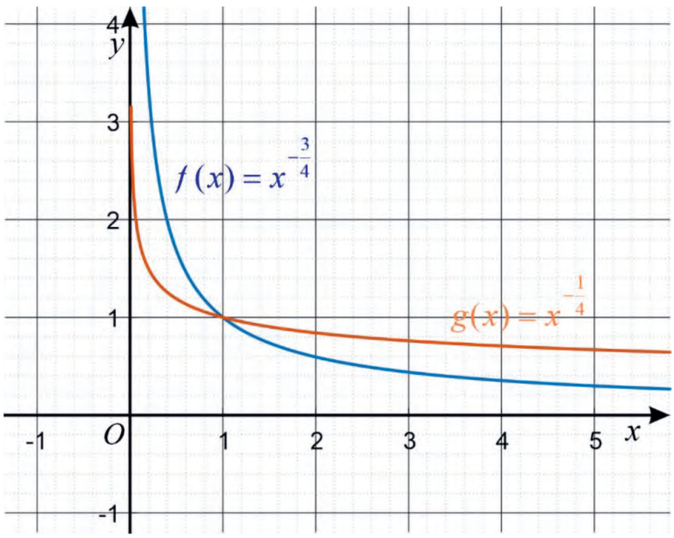


Figure 1.32: Graphs of $f(x) = x^{-\frac{1}{2}}$ and $g(x) = x^{-\frac{1}{4}}$

VII. Functions of the form $f(x) = x^{-\frac{1}{n}}$ for an even natural number n have similar properties and some of these common properties are:

a. Domain of the function is $(0, \infty)$.

b. Range of the function is $(0, \infty)$.

c. The graph of the function is a curve that has the same shape as the curves in **Figure 1.33.**

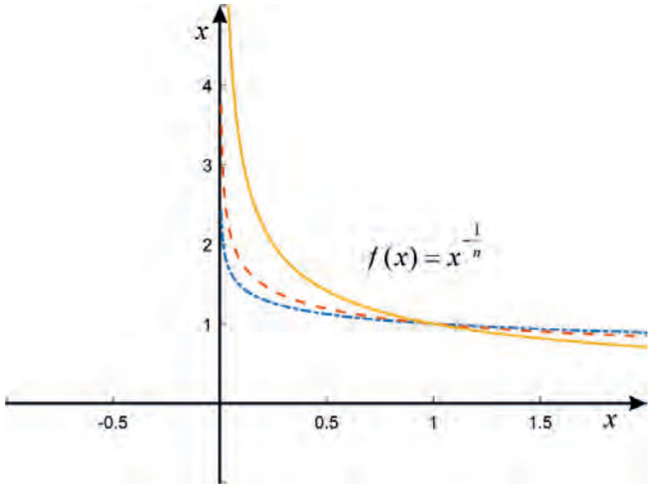


Figure 1.33: Graph of $f(x)=x^{-\frac{1}{n}}$ for an even natural number n.

Example 9

Find the domain and range of each of the following functions and sketch their graphs.

a. $f(x)=x^{-\frac{1}{3}}$ b. $g(x)=x^{-\frac{1}{5}}$

Solution

a. Domain of f is $\mathbb{R}\backslash\{0\}$ and Range of f is $\mathbb{R}\backslash\{0\}$.

b. Domain of g is $\mathbb{R}\backslash\{0\}$ and Range of g is $\mathbb{R}\backslash\{0\}$.

To draw the graphs of both functions f and g, use the table given below.

x	-32	$\cdots$	-8	$\cdots$	-1	1	$\cdots$	8	$\cdots$	32
$f(x)$	$-\frac{1}{2\sqrt[3]{4}}$	$\cdots$	-0.5	...	-1	1	$\cdots$	0.5	$\cdots$	$\frac{1}{2\sqrt[3]{4}}$
$g(x)$	-0.5	$\cdots$	$-\frac{1}{\sqrt[5]{8}}$	$\cdots$	-1	1	$\cdots$	$\frac{1}{\sqrt[5]{8}}$	$\cdots$	0.5

Then, using the values in the table, the graphs of f and g are given in **Figure 1.34.**

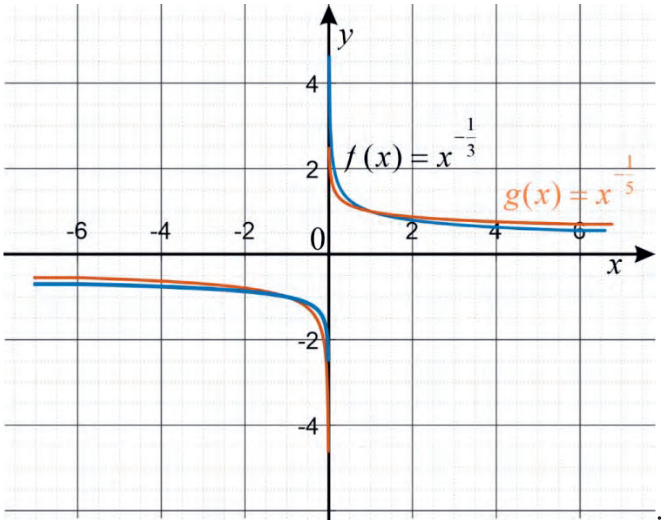


Figure 1.34: Graphs of $f(x)=x^{-\frac{1}{3}}$ and $g(x)=x^{-\frac{1}{5}}$

VIII. Functions of the form $f(x) = x^{-\frac{1}{n}}$ for an odd natural number n have similar properties and some of these common properties are:

a. Domain of the function is $\mathbb{R}\setminus\{0\}$.

b. Range of the function is $\mathbb{R}\setminus\{0\}$.

c. The graph of the function is a curve that has the same shape as the curves in **Figure 1.35.**

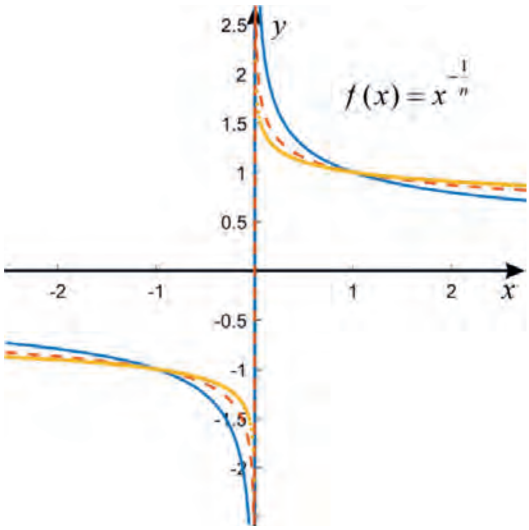


Figure 1.35: Graph of $f(x)=x^{-\frac{1}{n}}$ for an odd natural number n.

Exercise 1.12

For each of the following functions, find the domain, range and sketch the graph.

a. $f(x) = x^{-\frac{1}{6}}$ b. $g(x) = x^{-\frac{1}{8}}$ c. $h(x) = x^{-\frac{1}{7}}$ d. $l(x) = x^{-\frac{1}{9}}$

Basic Properties of Power Functions (5)

Example 10

Find the domain and range of each of the following functions and sketch their graphs.

a. $f(x) = x^{-\frac{3}{2}}$ b. $g(x) = x^{-\frac{2}{3}}$

Solution

a. $\text{Dom}(f) = (0, \infty)$ and $\text{Ran}(f) = (0, \infty)$.

b. $\text{Dom}(g) = \mathbb{R}\backslash\{0\}$ and $\text{Ran}(g) = (0, \infty)$.

To draw the graphs of both functions f and g, use the tables given below,

x	$\frac{1}{2}$	1	4	9
$f(x)$	$\sqrt{8}$	1	$\frac{1}{8}$	$\frac{1}{27}$

x	-8	-1	1	8
$g(x)$	$\frac{1}{4}$	1	1	$\frac{1}{4}$

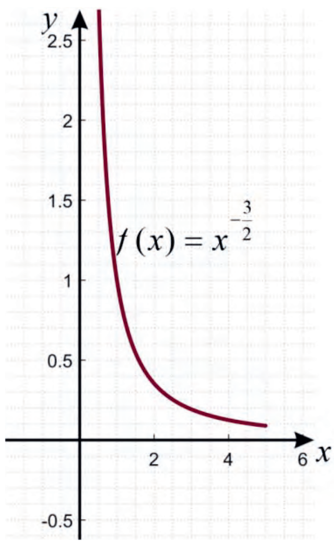


Figure 1.36: Graph of $f(x) = x^{-\frac{3}{2}}$

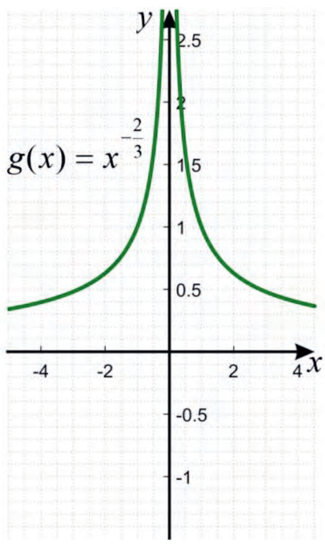


Figure 1.37: Graph of $g(x) = x^{-\frac{2}{3}}$

IX. Functions of the form $f(x) = x^{-\frac{m}{n}}$ for an odd natural number m and an even natural number n have similar properties and some of these common properties are:

a. Domain of the function is $(0, \infty)$.

b. Range of the function is $(0, \infty)$.

c. The graph of the function is a curve that has the same shape as the curves in **Figure 1.38.**

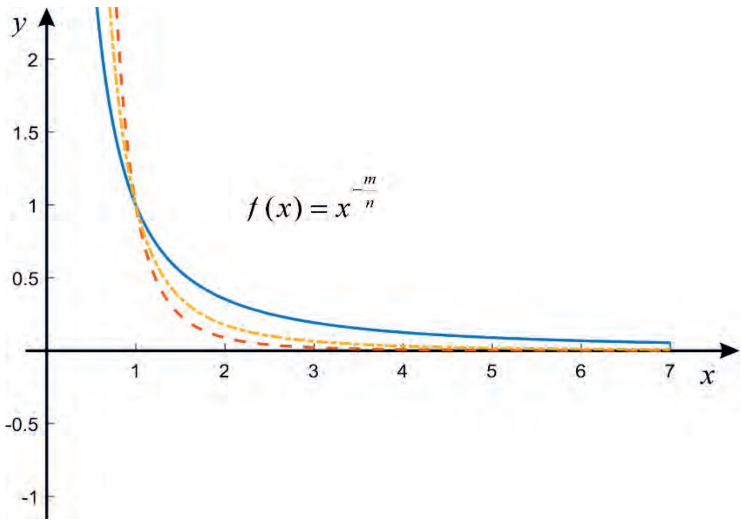


Figure 1.38: Graph of $f(x) = x^{-\frac{m}{n}}$, m is odd and n is even natural numbers.

X. Functions of the form $f(x) = x^{-\frac{m}{n}}$ for an even natural number m and an odd natural number n have similar properties and some of these common properties are:

a. Domain of the function is $\{x \in \mathbb{R} : x \neq 0\}$.

b. Range of the function is $(0, \infty)$.

c. The graph of the function is given in **Figure 1.39**.

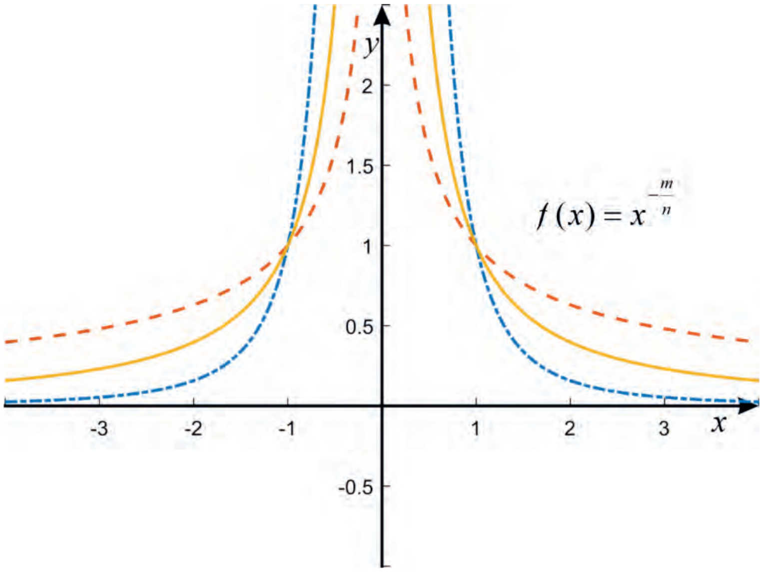


Figure 1.39: Graph of $f(x) = x^{-\frac{m}{n}}$, m is even and n is odd.

Exercise 1.13

For each of the following functions, find the domain, range and sketch the graph.

a. $f(x) = x^{-\frac{5}{4}}$

b. $g(x) = x^{-\frac{6}{5}}$

c. $h(x) = x^{-\frac{7}{6}}$

d. $l(x) = x^{-\frac{6}{7}}$

1.3.2 Modulus (Absolute Value) Function

Revision of Absolute Values

Activity 1.7

a. Find the distance from the origin to the point represented by -3 on the number line.

b. Find the distance from the origin to the point represented by 3 on the number line.

c. Find the distance from origin to the point represented by 0 on the number line.

From your response in Activity 1.7, observe that points represented by −3 and 3 on a number line are both 3 units from the origin, as in Figure 42 and the distance from the origin to itself is 0.

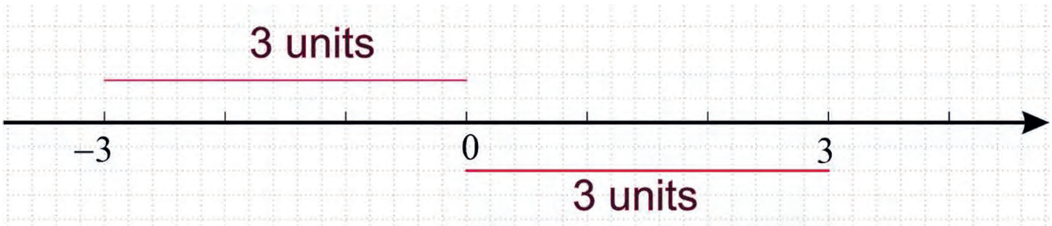


Figure 1.40

Distance on a number line is defined using the concept of absolute value.

Definition 1.5

Given a real number a, the absolute value (modulus) of a, denoted by $|a|$, is the nonnegative real number defined by

$$|a| = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

Example 1

Find the absolute value of each of the following numbers.

a. 6 b. 0 c. −5

Solution

a. $|6| = 6$, because $6 > 0$.

b. $|0| = 0$, by the definition of absolute value.

c. $|-5| = -(-5) = 5$, because $-5 < 0$.

Properties of Absolute Value

Activity 1.8

a. Find the absolute value of each of the numbers 4 and -4.

b. Compare the two results in (a).

From your responses in Activity 1.8, observe that $|4|$ and $|-4|$ are equal.

Thus we have the following general property of absolute value.

Property I

For any real number a, $|a| = |-a|$.

Activity 1.9

1. If $a = 3$ and $b = 4$, then
   a. find both $|ab|$ and $|a||b|$;
   b. compare $|ab|$ and $|a||b|$;
   c. find both $\left|\frac{a}{b}\right|$ and $\frac{|a|}{|b|}$;
   d. compare $\left|\frac{a}{b}\right|$ and $\frac{|a|}{|b|}$
   e. find both $|a+b|$ and $|a|+|b|$
   f. Compare $|a+b|$ and $|a|+|b|$
2. If $a = -3$ and $b = -4$, then
   a. find both $|ab|$ and $|a||b|$;
   b. compare $|ab|$ and $|a||b|$;
   c. find both $\left|\frac{a}{b}\right|$ and $\frac{|a|}{|b|}$;
   d. compare $\left|\frac{a}{b}\right|$ and $\frac{|a|}{|b|}$
   e. find both $|a+b|$ and $|a|+|b|$
   f. Compare $|a+b|$ and $|a|+|b|$
3. If $a = 3$ and $b = -4$, then
   a. find both $|ab|$ and $|a||b|$;
   b. compare $|ab|$ and $|a||b|$;
   c. find both $\left|\frac{a}{b}\right|$ and $\frac{|a|}{|b|}$;
   d. compare $\left|\frac{a}{b}\right|$ and $\frac{|a|}{|b|}$
   e. find both $|a+b|$ and $|a|+|b|$
   f. Compare $|a+b|$ and $|a|+|b|$

From Activity 1.9, you have observed the following properties of absolute values.

Property II

For any two real numbers a and b,

i. $|ab| = |a||b|$ ii. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ for $b \neq 0$.

iii. $|a + b| \leq |a| + |b|$ (Triangular Inequality).

Activity 1.10

Let $a = \frac{2}{5}$ and $b = -3$. Then

a. Find both $|a|$ and $|b|$.

b. Compare $|a|$ with both $\frac{2}{5}$ and $-\frac{2}{5}$.

c. Compare $|b|$ with 3 and -3.

d. Find both $|-a|$ and $|-b|$.

From your responses in Activity 1.10, observe that $\left|\frac{2}{5}\right| \geq \frac{2}{5}$ and $\left|\frac{2}{5}\right| \geq -\frac{2}{5}$ and also $|-3| \geq 3$ and $|-3| \geq -3$,

Thus, we have the following properties of absolute value.

Property III

a. For any real number a, $|a| \geq a$ and $|a| \geq -a$.

b. For any given positive real number a, $|a| = a$ and $|-a| = -(-a) = a$.
Thus, $|x| = a$ if and only if $x = a$ or $x = -a$.

c. For any real number x, $|x|$ can never be negative, as it is the distance from the origin to the point represented by x on the number line. Thus, $|x| \geq 0$.

Example 2

Solve each of the following equations.

a. $|x| = 7$ b. $|x| = 0$ c. $|x| = -3$ d. $|2x + 6| = 10$

Solution

a. If $|x| = 7$, then $x = 7$ or $x = -7$

b. If $|x| = 0$, then $x = 0$.

c. If $|x| = -3$, then there is no number for x that is a solution for this equation.

d. If $|2x + 6| = 10$, then $2x + 6 = 10$ or $2x + 6 = -10$.

This implies $2x = 10 - 6 \Rightarrow 2x = 4 \Rightarrow x = 2$ or

$2x + 6 = -10 \Rightarrow 2x = -10 - 6 \Rightarrow 2x = -16 \Rightarrow x = -8$.

Thus, the values of x that make the statement $|2x + 6| = 10$ true are 2 and -8.

Exercise 1.14

1. Find the absolute value of each of the following numbers.

   a. $-\frac{2}{5}$ b. $\frac{1}{7}$ c. $-\frac{5}{4}$

2. Compare the absolute values of each of the following pairs of numbers.

   a. -4.5 and 3.75 b. $\frac{2}{9}$ and $-\frac{5}{12}$

3. Solve each of the following equations.

   a. $|x| = 5$ b. $|x - 1| = 0$ c. $|3x + 6| = 9$ d. $|x + 1| = -2$

Modulus Functions

Definition 1.6

The function $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = |x|$ is called the absolute value (or modulus) function.

Example 3

Given $f(x) = |x|$, determine the values of each of the following.

$f(-1)$, $f(0)$ and $f(1)$

Solution

a. $f(-1) = |-1| = -(-1) = 1$;

b. $f(0) = |0| = 0$;

c. $f(1) = |1| = 1$.

Note

For any real number x, $|x|$ is defined and $|x| \geq 0$. Thus,

a. the domain of $f(x) = |x|$ is $\mathbb{R}$.

b. the range of $f(x) = |x|$ is $[0, \infty)$.

Example 4

Draw the graph of the function $f(x) = |x|$.

Solution

Consider the following table of values of f.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	4	3	2	1	0	1	2	3	4

Then using the table, you can draw the graph of the absolute value function $f(x) = |x|$ on the coordinate plane, as in **Figure 1.41.**

Observe that, $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.

So the graph of $f(x) = |x|$ is the graph of $y = x$ for $x \geq 0$ and the graph of $y = -x$ for $x < 0$.

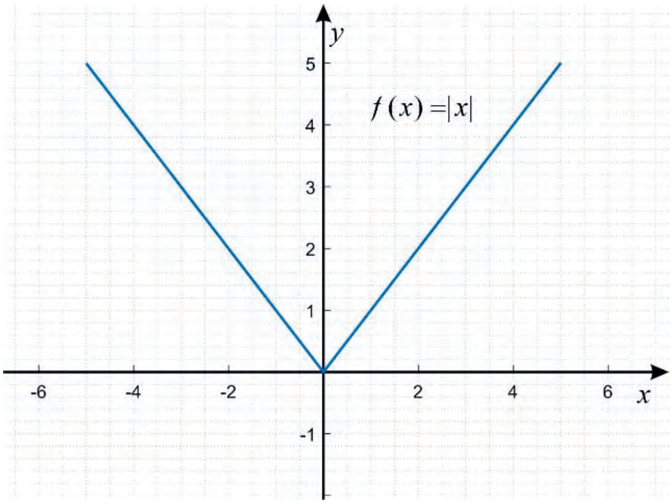


Figure 1.41: Graph of $f(x) = |x|$.

Observe that the graph of the absolute function $f(x) = |x|$

a. is a continuous curve; that is, it has no break;

b. passes through $(0, 0)$ and has a sharp corner at $(0, 0)$;

c. is symmetric with respect to y-axis; that is, the part of the graph of the function for $x > 0$ is the image of the reflection of the graph of the function for $x < 0$ with respect to the y −axis.

Example 5

Find the domain, range and sketch the graph of the function $f(x) = -|x|$

Solution

As $f(x) = -|x|$ defined for all x in $\mathbb{R}$ and $-|x| \leq 0$ for all $x \in \mathbb{R}$,

a. $\text{Dom}(f) = \mathbb{R}$ b. $\text{Ran}(f) = (-\infty, 0]$.

To draw the graph of f consider the following table of values of the function:

x	-3	-2	-1	0	1	2	3
$f(x)$	-3	-2	-1	0	-1	-2	-3

Then, plotting the points given in the table and joining these points with proper curves gives us the graph of f, **Figure 1.42.**

Observe that $f(x) = -|x| = \begin{cases} -x \text{ if } x \geq 0 \\ -(-x) = x \text{ if } x < 0 \end{cases}$.

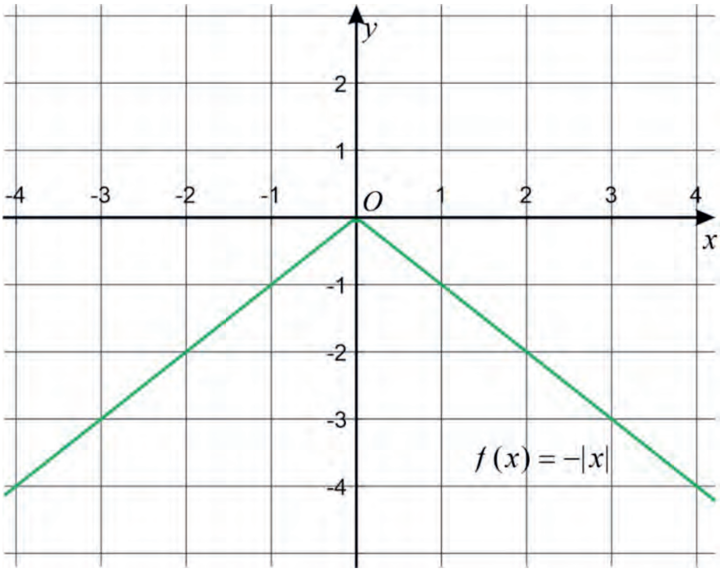


Figure 1.42: Graph of $f(x) = -|x|$

Example 6

Find the domain, range and sketch the graph of the function $f(x) = |x-1|$.

Solution

As $f(x) = |x-1|$ defined for all x in $\mathbb{R}$ and $|x-1| \geq 0$ for all $x \in \mathbb{R}$,

a. $\text{Dom}(f) = \mathbb{R}$ b. $\text{Ran}(f) = [0, \infty)$.

To draw the graph of f consider the following table of values of the function.

x	-3	-2	-1	0	1	2	3
$f(x)$	4	3	2	1	0	1	2

Then, plotting the points given in the table and joining these points with proper curves gives us the graph of f, **Figure 1.43.**

Observe that $f(x) = \begin{cases} x-1, \text{when } x \geq 1; \\ -x+1, \text{when } x < 1. \end{cases}$

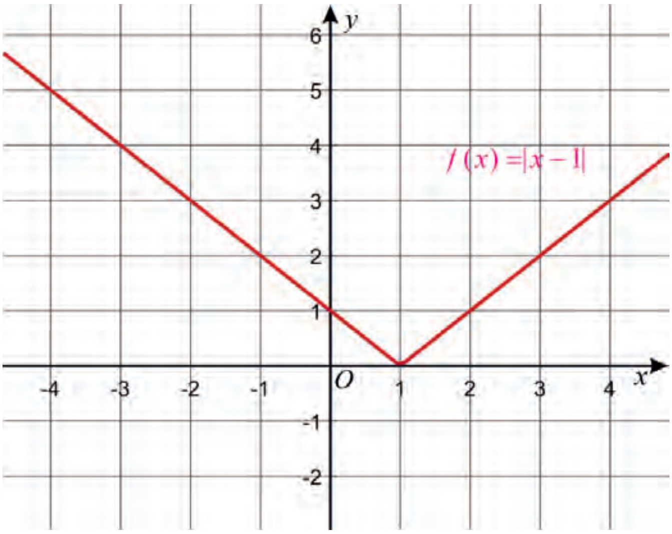


Figure 1.43: Graph of $f(x) = |x - 1|$

Example 7

Find the domain, range and then sketch the graph of the function $f(x) = |x| + 1$.

Solution

As $f(x) = |x| + 1$ defined for all x in $\mathbb{R}$ and $|x| + 1 \geq 1$ for all $x \in \mathbb{R}$,

a. $\text{Dom}(f) = \mathbb{R}$

b. $\text{Ran}(f) = [1, \infty)$.

To draw the graph of f consider the following table of values of the function:

x	-3	-2	-1	0	1	2	3
$f(x)$	4	3	2	1	2	3	4

Then, plotting the points given in the table and joining these points with proper curves gives you the graph of f, **Figure 1.44.**

Observe that when $x \geq 0, f(x) = x + 1$ and when $x < 0, f(x) = -x + 1$.

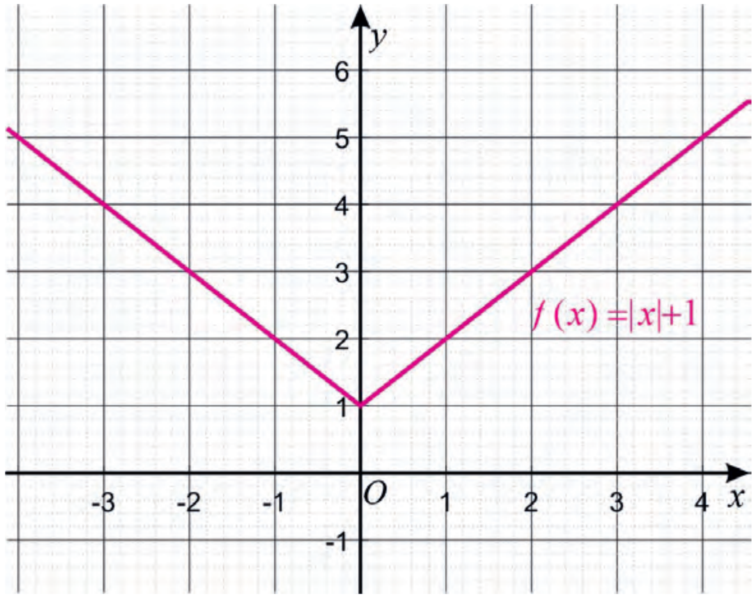


Figure 1.44: Graph of $f(x) = |x| + 1$

Example 8

Find the domain, range and sketch the graph of the function $f(x) = |2x|$.

Solution

As $f(x) = |2x|$ defined for all x in $\mathbb{R}$ and $|2x| \geq 0$ for all $x \in \mathbb{R}$,

a. $\text{Dom}(f) = \mathbb{R}$ b. $\text{Ran}(f) = [0, \infty)$.

To draw the graph of f consider the following table of values of the function:

x	-3	-2	-1	0	1	2	3
$f(x)$	6	4	2	0	2	4	6

Then, plotting the points given in the table and joining these points with proper curves gives you the graph of f, **Figure 1.45.**

Observe that when $x \geq 0, f(x) = 2x$ and when $x < 0, f(x) = -2x$.

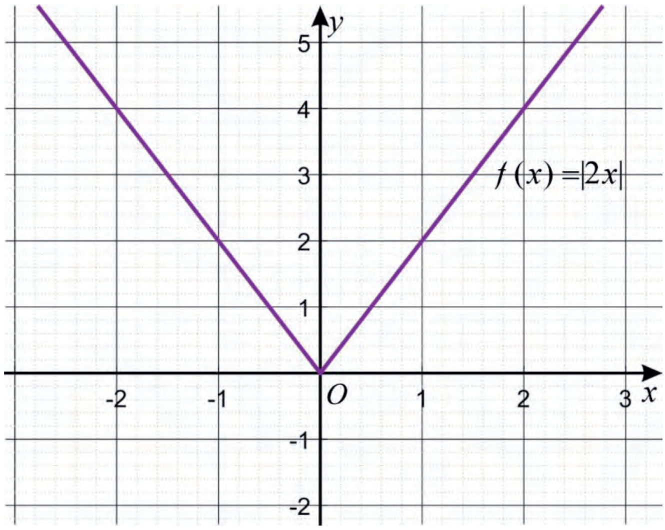


Figure 1.45: Graph of $f(x)=|2x|$

Exercise 1.15

Find the domain, range and sketch the graph of each of the following functions.

a. $f(x)=|x+1|$ b. $g(x)=|x|-1$ c. $h(x)=2-|x|$ d. $h(x)=|5x|$

1.3.3 Signum Function

Activity 1.11

Consider the function $f(x)=2$ for all $x\in(-\infty,0]$ and $g(x)=3$ for all $x\in(0,\infty)$.

a. Find the domains and ranges of both f and g.

b. Let $h(x)=f(x)$ for all $x\in(-\infty,0]$ and $h(x)=g(x)$ for all $x\in(0,\infty)$. Find the domain and range of h.

c. Sketch the graphs of f, g and h.

From your responses in Activity 1.11, you have observed that the functions f and g are constant functions and the function h is defined piecewise, one value for $x\leq 0$ and another value for $x>0$.

One of the common piecewise functions is the signum function and its definition is given below.

Definition 1.7

The function defined by $f(x)=\begin{cases}1 & \text{for } x>0;\\ 0 & \text{for } x=0; \\ -1 & \text{for } x<0.\end{cases}$ and is called the signum function denoted by $sgn\ x$ and read as signum x.

That is, $sgn\,x=\begin{cases}1 & \text{for } x>0;\\ 0 & \text{for } x=0; \text{ and}\\ -1 & \text{for } x<0.\end{cases}$

Note

The function $sgn\ x=\begin{cases}1 & \text{for } x>0\\ 0 & \text{for } x=0\\ -1 & \text{for } x<0\end{cases}$ has the following properties.

a. Its domain is $\mathbb{R}$

b. Its range is the set $\{-1,0,1\}$

For any real number x, $\frac{|x|}{x}=\begin{cases}\frac{x}{x}=1 & \text{for } x>0\\ \text{undefined} & \text{for } x=0\\ \frac{-x}{x}=-1 & \text{for } x<0\end{cases}$

Therefore, we can define the signum function using absolute value as:

$$sgn\ x=\begin{cases}\frac{|x|}{x} & \text{for } x\neq 0\\ 0 & \text{for } x=0\end{cases}$$

Now we draw the graph of the signum function using the table given below.

x	-4	-3	-2	-1	0	1	2	3	4
$sgn\ x$	-1	-1	-1	-1	0	1	1	1	1

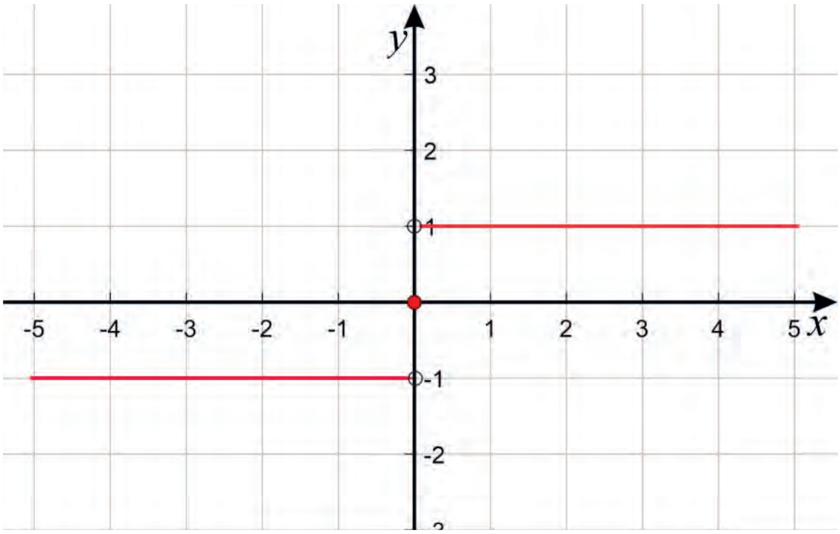


Figure 1.46: Graph of $f(x) = \text{sgn}\, x$

Example 1

Determine the domain and range of the function $f(x) = sgn\ x + 2$ and sketch its graph.

Solution

From the definition of the signum function:

$$f(x) = \text{sgn}\, x + 2 = \left\{\begin{array}{ll} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{array}\right\} + 2 = \left\{\begin{array}{ll} 3 & \text{for } x > 0 \\ 2 & \text{for } x = 0 \\ 1 & \text{for } x < 0 \end{array}\right.$$

Then the domain of f is $\mathbb{R}$ and its range is the set $\{1,2,3\}$.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	1	1	1	1	2	3	3	3	3

Then, using the table, the graph of f is sketched in **Figure 1.47.**

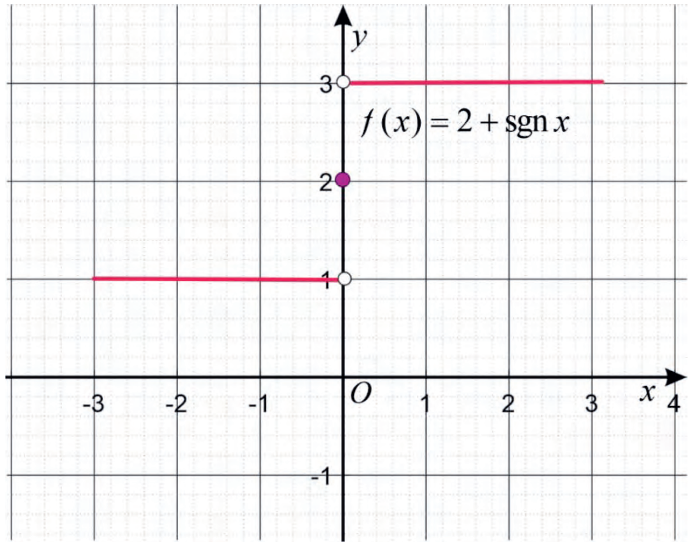


Figure 1.47: Graph of $f(x) = 2 + sgn\ x$

Example 2

Determine the domain and range of the function $f(x) = 4sgn\ x$ and sketch its graph.

Solution

From the definition of the signum function:

$$f(x) = 4\text{sgn}\, x = 4 \times \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases} = \begin{cases} 4 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -4 & \text{for } x < 0 \end{cases}$$

Then, the domain of f is $\mathbb{R}$ and its range is the set $\{-4, 0, 4\}$.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-4	-4	-4	-4	0	4	4	4	4

Then, using the above table, the graph of f is sketched in **Figure 1.48.**

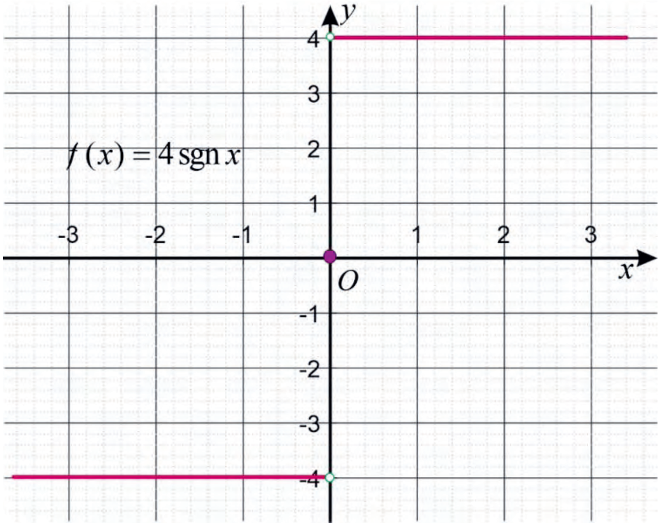


Figure 1.48: Graph of $f(x) = 4sgnx$

Exercise 1.16

Find the domain and range of each of the following functions and sketch its graph.

a. $f(x)=2\text{sgn}\,x$

b. $g(x)=-3\text{sgn}\,x$

c. $h(x)=1+\text{sgn}\,x$

d. $l(x)=3-\text{sgn}\,x$

1.3.4 The Greatest Integer (Floor or Step) Function

Greatest Integer Function

Activity 1.12

For each of the following numbers, find the greatest integer that is less than or equal to the given number.

a. 3.7 b. −1.9 c. 6 d. −4

From Activity 1.12, you have observed that you can find the greatest integer that is less than or equal to a given number, for example, the greatest integer less than or equal to -1.9 is -2.

Definition 1.8

The function defined by $f(x)$ =the greatest integer that is less than or equal to x, denoted by $f(x) = \lfloor x \rfloor$ is called the Greatest Integer (Floor or Step) Function.

Example 1

Evaluate the following.

a. $\lfloor 10.7 \rfloor$ b. $\lfloor -12.4 \rfloor$ c. $\lfloor -18 \rfloor$

Solution

a. The greatest integer that is less than or equal to 10.7 is 10, that is $\lfloor 10.7 \rfloor = 10$.

b. The greatest integer that is less than or equal to -12.4 is -13, that is $\lfloor -12.4 \rfloor = -13$.

c. The greatest integer that is less than or equal to -18 is -18, that is $\lfloor -18 \rfloor = -18$.

Exercise 1.17

Evaluate each of the following.

a. $\lfloor \pi \rfloor$ c. $\lfloor \sqrt{2} \rfloor$ e. $\left\lfloor -\frac{23}{25} \right\rfloor$

b. $\left\lfloor \frac{23}{25} \right\rfloor$ d. $\lfloor -\pi \rfloor$ f. $\lfloor -\sqrt{2} \rfloor$

Graph of the Greatest Integer Function

Given the function $f(x) = \lfloor x \rfloor$:

a. Dom(f) is the set of all real numbers.

b. Ran(f) is the set of all integers.

To understand the behavior of this function in terms of a graph, let us construct a table of values as in the following table:

x	$-3 \leq x < -2$	$-2 \leq x < -1$	$-1 \leq x < 0$	$0 \leq x < 1$	$1 \leq x < 2$	$2 \leq x < 3$
$y = \lfloor x \rfloor$	-3	-2	-1	0	1	2

The table shows us that the function increases to the next higher integer any time the x-value becomes an integer and the graph of the function $y = \lfloor x \rfloor$ is given in **Figure 1.49**.

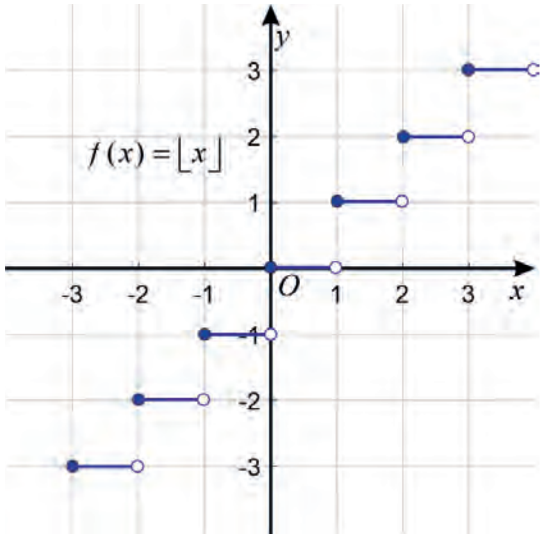


Figure 1.49: Graph of $f(x) = \lfloor x \rfloor$

Example 2

Find the domain, range and sketch the graph of $f(x) = \lfloor 2x \rfloor$.

Solution

The function $f(x) = \lfloor 2x \rfloor$ is defined for all real numbers and the set of its functional values is the set of integers. Therefore,

a. Dom(f) is the set of all real numbers.

b. Ran(f) is the set of all integers.

x	$-1 \leq x < -0.5$	$-0.5 \leq x < 0$	$0 \leq x < 0.5$	$0.5 \leq x < 1$	$1 \leq x < 1.5$
$y = \lfloor 2x \rfloor$	-2	-1	0	1	2

Using the points in the table, the graph of the function $f(x) = \lfloor 2x \rfloor$ is sketched in **Figure 1.50**.

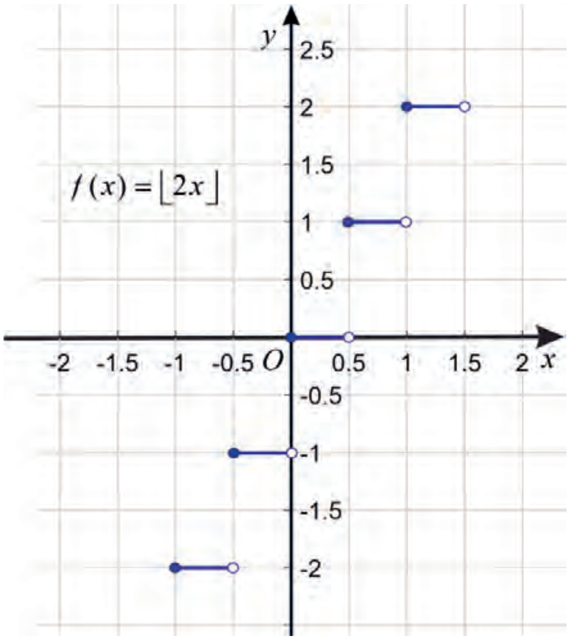


Figure 1.50: Graph of $f(x)=\lfloor 2x \rfloor$.

Example 3

Find the domain, range and then sketch the graph of $f(x)=\lfloor x+1 \rfloor$.

Solution

The function $f(x)=\lfloor x+1 \rfloor$ is defined for all real numbers and the set of its functional values is the set of integers. Therefore,

a. Dom(f) is the set of all real numbers.

b. Ran(f) is the set of all integers.

x	$-2 \leq x < -1$	$-1 \leq x < 0$	$0 \leq x < 1$	$1 \leq x < 2$	$2 \leq x < 3$
$y = [x+1]$	-1	0	1	2	3

Using the points in the above table, the graph of the function $f(x) = \lfloor x+1 \rfloor$ is sketched in **Figure 1.51**.

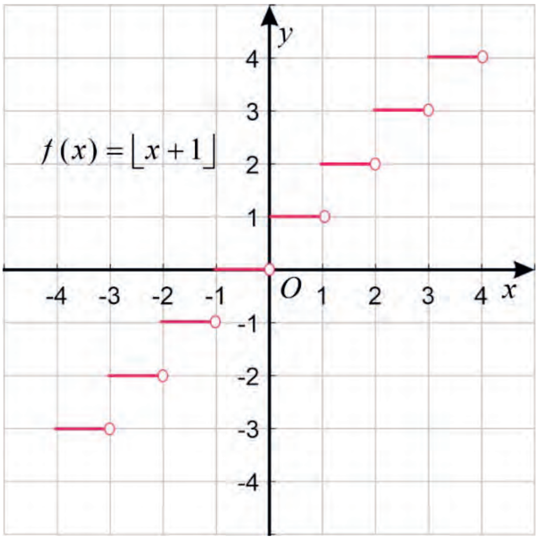


Figure 1.51: Graph of $f(x) = \lfloor x + 1 \rfloor$.

Example 4

Find the domain, range and sketch the graph of $f(x) = 3\lfloor x \rfloor$.

Solution

The function $f(x) = 3\lfloor x \rfloor$ is defined for all real numbers and the set of its functional values is the set of all integer multiples of 3. Therefore,

a. Dom(f) is the set of all real numbers.

b. Ran(f) is the set of all integer multiples of 3.

x	$-2 \le x < -1$	$-1 \le x < 0$	$0 \le x < 1$	$1 \le x < 2$	$2 \le x < 3$
$y = 3\lfloor x \rfloor$	-6	-3	0	3	6

Using the points in the table, the graph of the function $f(x) = 3\lfloor x \rfloor$ is sketched in **Figure 1.52**.

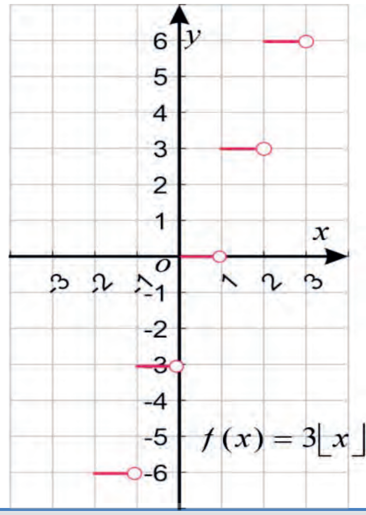


Figure 1.52: Graph of $f(x) = 3\lfloor x \rfloor$

Exercise 1.18

Sketch the graph of each of the following functions.

a. $f(x) = \lfloor 4x \rfloor$ b. $g(x) = 2\lfloor x \rfloor$ c. $h(x) = \lfloor x+3 \rfloor$

Equations involving the Greatest Integer Function:

When working with equations that involve the greatest integer function, we use the following formula: $\lfloor x \rfloor = m$, where m is an integer if and only if $m \le x < m+1$.

For example, $\lfloor x \rfloor = 6$ if and only if $6 \le x < 7$.

Example 5

Solve the equation $\lfloor x + 3 \rfloor = 10$.

Solution

First, rewrite the equation using the inequalities $10 \le x + 3 < 11$.

Then, solve the inequalities: $10 \le x + 3 < 11 \Rightarrow 10 - 3 \le x < 11 - 3$

$$\Rightarrow 7 \le x < 8$$

Therefore, the solution set is the interval $[7,8)$.

Example 6

Solve the equation $\lfloor 1+\lfloor x\rfloor\rfloor = 5$.

Solution

First let us replace $\lfloor x\rfloor$ with z. This is called a "change of variable" and it will make the equation easier to work with.

$$\lfloor 1+\lfloor x\rfloor\rfloor = 5 \Longrightarrow \lfloor 1+z\rfloor = 5$$

Then, let us solve the equation, $\lfloor 1+z\rfloor = 5$.

Replace the given equation with the inequalities, $5 \le 1+z < 6$

Solve the inequality

$$5 \le 1+z < 6 \Longrightarrow 4 \le z < 5 \Longrightarrow 4 \le \lfloor x\rfloor < 5$$

Since $\lfloor x\rfloor$ is an integer, the only way to satisfy the above inequalities is for $\lfloor x\rfloor = 4$.

Let us determine the value of x.

Again, using the inequalities, we know $4 \le x < 5$

Therefore, the solution set of the given equation is the interval $[4,5)$.

Exercise 1.19

Solve each of the following equations.

a. $\lfloor x+4\rfloor = 6$

b. $\lfloor 3+\lfloor x\rfloor\rfloor = 5$

1.4 Composition of Functions

Activity 1.13

Consider the functions $f(x)=x+3$ and $g(x)=x^2$. What result will you have if

a. you substitute $x+3$ to x in $g(x)$?

b. you substitute x^2 to x in $f(x)$?

From your responses in Activity 1.13, observe that:

a. you get $g(x+3)=(x+3)^2$ and

b. you get $f(x^2) = x^2 + 3$.

$g(x+3)=(x+3)^2$ is the result of substituting $f(x) = x + 3$ into $g(x)$.

That is, $g(x+3)=g(f(x))$, which is called the composition g by f.

Similarly $f(x^2) = x^2 + 3$ is the result of substituting $g(x)=x^2$ into $f(x)$.

That is, $f(x^2)=f(g(x))$, which is called the composition f by g.

Note that, $g(f(x))$ is denoted by $(g \circ f)(x)$ and $f(g(x))$ is denoted by $(f \circ g)(x)$.

Definition 1.9

Let $f: A \to B$ and $g: B \to C$ be functions, Then the function h defined by $h(x)=g(f(x))$ for all $x \in A$ and $f(x)$ is in the domain of g, is called the composition of g by f, denoted by $g \circ f$. That is, $(g \circ f)(x)=g(f(x))$, for all $x \in A$ and $f(x)$ is in the domain of g.

Example 1

Let $f=\{(-1,1),(0,2),(1,-1),(2,5)\}$ and

$g=\{(-1,2),(0,0),(1,3),(2,1),(3,4),(4,2),(5,5)\}$.

Then, find $g \circ f$ and $f \circ g$.

Solution

$(g \circ f)(-1)=g(f(-1))=g(1)=3$,

$$(g \circ f)(0)=g(f(0))=g(2)=1,$$

$(g \circ f)(1)=g(f(1))=g(-1)=2$,

$$(g \circ f)(2)=g(f(2))=g(5)=5.$$

Therefore, $g \circ f=\{(-1,3),(0,1),(1,2),(2,5)\}$.

$(f \circ g)(-1) = f(g(-1)) = f(2) = 5,$

$(f \circ g)(0) = f(g(0)) = f(0) = 2$

$(f \circ g)(2) = f(g(2)) = f(1) = -1$

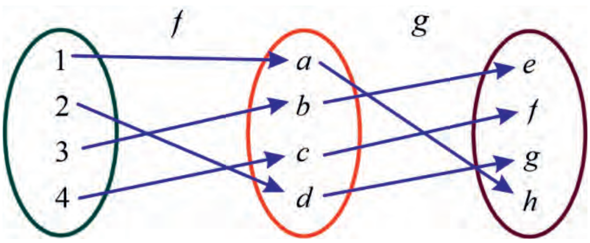
$(f \circ g)(4) = f(g(4)) = f(2) = 5.$

Therefore, $f \circ g = \{(-1,5),(0,2),(2,-1),(4,5)\}$

Observe that, $(f \circ g)(1)$ is not defined, because $g(1) = 3$ and 3 is not in the domain of f.

Example 2

Consider functions f and g given in the following diagram.



Then, determine $g \circ f$.

Solution

First observe that $f = \{(1,a),(2,d),(3,b),(4,c)\}$ and $g = \{(a,h),(b,e),(c,f),(d,g)\}$.

Thus,

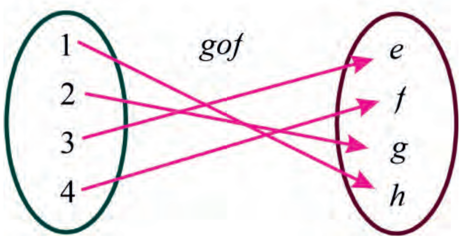
i. $(g \circ f)(1) = g(f(1)) = g(a) = h$

ii. $(g \circ f)(2) = g(f(2)) = g(d) = g$

iii. $(g \circ f)(3) = g(f(3)) = g(b) = e$

iv. $(g \circ f)(4) = g(f(4)) = g(c) = f$

Therefore, $g \circ f = \{(1,h),(2,g),(3,e),(4,f)\}$.



Example 3

Let $f(x)=x+1$ and $g(x)=x^2-2$. Then find $f\circ g$ and $g\circ f$.

Solution

a. $(f\circ g)(x)=f(g(x))=f(x^2-2)=(x^2-2)+1=x^2-1$

b. $(g\circ f)(x)=g(f(x))=g(x+1)=(x+1)^2-2=x^2+2x-1$

Example 4

Let $f(x)=x^2+1$ and $g(x)=\sqrt{x}$. Then find

a. $(g\circ f)(2)$

b. $(g\circ f)(x)$

Solution

a. $(g\circ f)(2)=g(f(2))=g(5)=\sqrt{5}$

b. $(g\circ f)(x)=g(f(x))=g(x^2+1)=\sqrt{x^2+1}$.

Note

For two functions f and g, the domain of $g\circ f$ is the set of all x in the domain of f, such that $(g\circ f)(x)=g(f(x))$ is defined.

Exercise 1.20

1. Let $f(x)=x^2+2x$ and $g(x)=3x-5$. Then find

   a. $(f\circ g)(3)$ b. $(g\circ f)(3)$

   c. $(g\circ f)(0)$ d. $(f\circ g)(0)$

2. Let $f(x)=\sqrt{x}$ and $g(x)=2x-5$. Then determine

   a. $(g\circ f)(x)$ b. $(f\circ g)(x)$

   c. Domain of $g\circ f$ d. Domain of $f\circ g$

1.5 Inverse Functions and Their Graphs

In Section 1.1, you have learned about inverses of relations. In this section, you will learn about the inverses of functions. As functions are special types of relations, the idea of inverse of a function is a continuation of the idea of inverse of a relation, but the only problem here is that the inverse of a function may not be a function.

Inverse Function

Activity 1.14

1. Find the inverse of each of the following functions.

   a. $f=\{(1,2),(2,3),(3,4),(4,5)\}$ c. $h=\{(x,y)\mid x,y\in\mathbb{R}, y=x^2+1\}$

   b. $g=\{(-1,a),(0,b),(3,4),(4,5)\}$ d. $l=\{(x,y)\mid x,y\in\mathbb{R}, y=2x+1\}$

2. From (1) above, which inverses are functions and which are not? (Give your reasons).

From your responses in Activity 1.14, you have observed that the inverse of a function may or may not be a function.

Note

a. Given a function f, if its inverse is also a function, then we say that f is invertible and we denote its inverse by f^{-1}.

b. Given a function $y = f(x)$, if f is invertible then to find its inverse;

I. interchange x and y in the expression $y = f(x)$, that is., $x = f(y)$;

II. solve y in terms of x, (if possible);

III. write f^{-1} as $y = f^{-1}(x)$.

Example 1

Find the inverse of each of the following functions.

a. $f(x) = 3x + 2$ b. $g(x) = 6 - 2x$ c. $h(x) = \frac{x+3}{2x-4}, x \neq 2$

Solution

a. $y = f(x) = 3x + 2$.

- Interchange y and x and obtain $x = 3y + 2$
- Solve y for x and obtain $y = \frac{1}{3}x - \frac{2}{3}$.
- Therefore, $f^{-1}(x) = \frac{1}{3}x - \frac{2}{3}$

b. $y = g(x) = 6 - 2x$

- Interchange y and x and obtain $x = 6 - 2y$
- Solve for y in terms of x and obtain $y = -\frac{1}{2}x + 3$.
- Therefore, $g^{-1}(x) = -\frac{1}{2}x + 3$

c. $y = h(x) = \frac{x+3}{2x-4}, x \neq 2$

- Interchange y and x in the formula and obtain $x = \frac{y+3}{2y-4}$

- Solve for y in terms of x and obtain $y=\dfrac{4x+3}{2x-1}$.
- Therefore, $h^{-1}(x)=\dfrac{4x+3}{2x-1}$ for $x\neq\dfrac{1}{2}$.

Note

There are functions whose inverses are not functions.

Example 2

Show that the inverse of the function $f(x) = x^3 - x + 1$ is not a function.

Solution

$f(-1)=(-1)^3-(-1)+1=1$ and $f(1)=1^3-1+1=1$.

This implies that $f(-1)=f(1)$.

Thus $(1,-1)$ and $(1,1)$ are elements of the inverse of f, but $-1\neq 1$.

Thus, the inverse of f is not a function.

Exercise 1.21

Find the inverses of each of the following functions.

a. $f(x)=2x+3$

b. $f(x)=-3x+2$

c. $f(x)=\dfrac{x+5}{3x-4}$ for $x\neq\dfrac{4}{3}$.

Identity Function

Definition 1.10

Given a set A, if $f:A\rightarrow A$ is a function given by $f(x)=x$ for all $x\in A$, then f is called the identity function on A and denoted by I.

Activity 1.15

Let $f = \{(1,2),(2,3),(3,4),(4,1)\}$ a function defined on a set $A = \{1,2,3,4\}$ and I be the identity on A as, $I = \{(1,1),(2,2),(3,3),(4,4)\}$.

Then determine

a. $I \circ f$　　b. $f \circ I$　　c. Compare $I \circ f$ and $f \circ I$.

In Activity 1.15, the two compositions, $f \circ I$ and $I \circ f$, are equal and both are equal to f.

Example 3

Let $f(x) = 2x - 3$ and $I(x) = x$. Then

$(I \circ f)(x) = I(f(x)) = I(2x-3) = 2x-3 = f(x)$ and

$(f \circ I)(x) = f(I(x)) = f(x) = 2x-3$.

This implies $I \circ f = f = f \circ I$.

Note

In general, if f is a function and I is the identity function, then $I \circ f = f = f \circ I$

Activity 1.16

Consider the function $f = \{(1,2),(2,3),(3,4),(4,5)\}$ from $A = \{1,2,3,4\}$ to $B = \{2,3,4,5\}$. Then find,

a. f^{-1}　　b. $f^{-1} \circ f$　　c. $f \circ f^{-1}$

In Activity 1.16, you can observe that $f^{-1} \circ f$ is the identity on A and $f \circ f^{-1}$ is the identity on B.

Note

Given a function f, a function g is the inverse of f if and only if $g \circ f = I$ and $f \circ g = I$.

Example 4

Let $f(x) = 3x - 1$ then $f^{-1}(x) = \frac{1}{3}x + \frac{1}{3}$.

- $(f \circ f^{-1})(x) = f(f^{-1}(x)) = f\left(\frac{1}{3}x + \frac{1}{3}\right) = 3\left(\frac{1}{3}x + \frac{1}{3}\right) - 1 = x = I(x)$ and
- $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(3x - 1) = \frac{1}{3}(3x - 1) + \frac{1}{3} = x = I(x)$.

This implies, $f \circ f^{-1} = I = f^{-1} \circ f$.

Example 5

Determine if each of the following pairs of functions are inverse of each other.

a. $f(x) = 2x - 5$ and $g(x) = -2x + 5$.

b. $f(x) = \frac{x+2}{x-2}$ and $g(x) = \frac{2x+2}{x-1}$.

Solution

Let us use the idea of composition to determine if f and g are inverses of each other.

a. $(f \circ g)(x) = f(g(x)) = f(-2x + 5) = 2(-2x + 5) - 5 = -4x + 5$ and

$(g \circ f)(x) = g(f(x)) = g(2x - 5) = -2(2x - 5) + 5 = -4x + 15$.

This implies $g \circ f \neq I$ and $f \circ g \neq I$. Thus g is not the inverse of f.

From this example, observe that $g(x) = -f(x)$ and hence $f^{-1} \neq -f$. The inverse of f is $f^{-1}(x) = \frac{1}{2}x + \frac{5}{2}$

Using your knowledge in finding the inverses of functions, you can find that,

b. $(g \circ f)(x) = g(f(x)) = g\left(\frac{x+2}{x-2}\right) = \frac{2\left(\frac{x+2}{x-2}\right)+2}{\left(\frac{x+2}{x-2}\right)-1} = \left(\frac{4x}{x-2}\right)\left(\frac{x-2}{4}\right) = x = I(x)$ and

similarly you can show that $(f \circ g)(x) = x = I(x)$.

Hence, g is an inverse for *f*, that is, $f^{-1}(x) = g(x)$.

Exercise 1.22

1. Let $f(x) = 3x - 1$ and $\mathrm{I}(x) = x$. Determine $\mathrm{I} \circ f$ and $f \circ \mathrm{I}$.
2. Let $f(x) = 2x + 4$ and $\mathrm{I}(x) = x$.
   a. Determine $f^{-1}(x)$.
   b. Show that $(f^{-1} \circ f)(x) = \mathrm{I}(x)$ and
   c. Show that $(f \circ f^{-1})(x) = \mathrm{I}(x)$.

Graphs of Inverse Functions

As functions are relations, from our discussions on graphs of inverse relations, recall that the graph of the inverse of a given function can be obtained by reflecting the graph of the function with respect to the line $y = x$.

Example 6

Draw the graph of the inverse of each of the following functions.

a. $f = \{(1,3),(2,1),(3,2),(4,4)\}$

b. $g(x) = 2x + 1$

Solution

a. $f^{-1} = \{(3,1),(1,2),(2,3),(4,4)\}$ and the graph of f^{-1} is obtained by reflecting the graph of f with respect to the line $y = x$, as given in **Figure 1.53.**

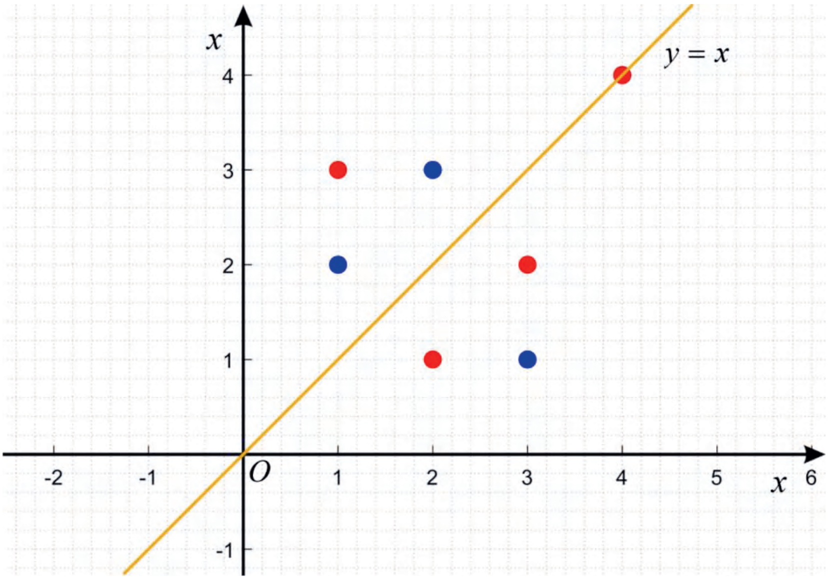


Figure 1.53: Graph of f and f^{-1}

b. Let $y = 2x + 1$. Then, interchanging x and y in the given equation gives us $x = 2y + 1$. Solving for y in terms of x gives you $y = \frac{1}{2}x - \frac{1}{2}$; that is, $g^{-1}(x) = \frac{1}{2}x - \frac{1}{2}$.

Therefore, the graph of g^{-1} is obtained by reflecting the graph of g with respect to the line $y = x$ as given in **Figure 1.54.**

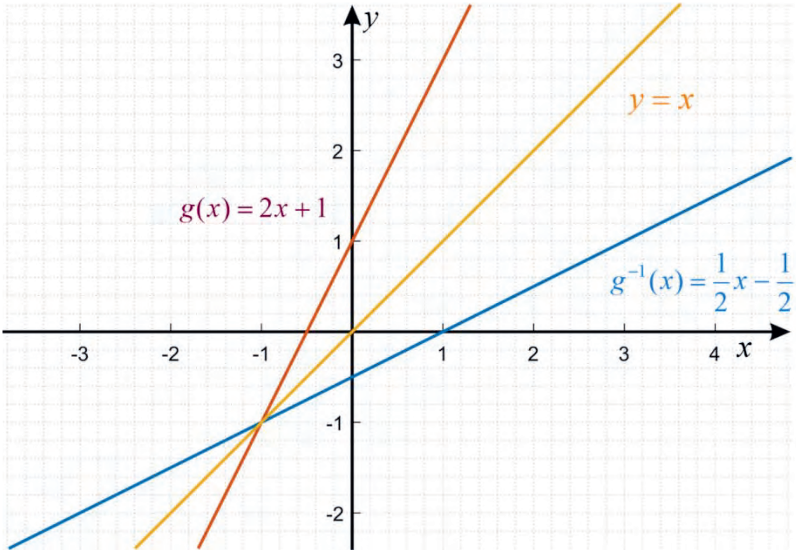


Figure 1.54 : Graphs of $g(x) = 2x + 1$ and $g^{-1}(x) = \frac{1}{2}x - \frac{1}{2}$

Exercise 1.23

1. Use compositions of functions to determine if each pairs of functions are inverses of each other.

   a. $f(x) = 8x$ and $g(x) = \frac{x}{8}$

   b. $f(x) = \frac{2}{3}x + 2$ and $g(x) = \frac{3}{2}x + 3$

   c. $f(x) = 5x - 7$ and $g(x) = \frac{x+5}{7}$

   d. $f(x) = \frac{x}{2+x}$ and $g(x) = \frac{2x}{1-x}$

2. Sketch the graphs of the functions and its inverse on the same coordinate plane for each of the following functions.

   a. $f(x) = x - 2$ b. $g(x) = 5x - 6$

1.6 Applications of Relations and Functions

In this section, different applications of relations and functions are given.

Example 1

Let $p+3q=30$ be an equation involving two variables p (price) and q (quantity). Indicate the meaningful domain and range of this function when

a. the price is considered as an independent variable;

b. the quantity is considered as an independent variable.

Solution

a. When price (p) is taken as an independent variable, you have $q(p)=10-\frac{1}{3}p$.

Domain: $0 \le p \le 30$ Range: $0 \le q \le 10$

b. When quantity (q) is taken as an independent variable, you have $p(q)=30-3q$.

Domain: $0 \le q \le 10$ Range: $0 \le p \le 30$

Example 2

A car rental company charges an initial fee, which is also called a flat fee of Birr 300 and an additional Birr 15 per kilometer to rent a van.

a. Write a function that approximates the cost y (in Birr) in terms of x, the number of kilometers driven.

b. How much would an 80 kilometer trip cost?

Solution

a. The total cost of renting a van is equal to the rate per kilometer times the number of kilometers driven plus the cost for the flat fee.

That is, $y=15x+300$

b. To calculate the cost of a 80 kilometers trip, substitute 80 for x in the equation:

$$y = 15x + 300$$

Thus, $y(80) = (15 \times 80) + 300 = 1500$.

Therefore, the cost of 80 kilometers trip is 1500 Birr.

Example 3

A man that buys and sells newspapers buys newspapers for price of Birr 6 per newspaper and sells newspapers at price of Birr 10 per newspaper. The unsold newspapers at the end of the day can be sold at 4 Birr per newspaper to a wastepaper dealer. If the man bought 300 newspapers, sold 200 newspapers and another 50 newspapers to a wastepaper dealer, what is the profit of the man in that day?

Solution

The overall profit to the man depends on the number of newspapers that he sells in relation to the number of newspapers he bought at the beginning of the day.

Purchase: 300 × 6 Birr = 1800 Birr

Sale: (200 × 10 Birr) + (50 × 4 Birr)=2000 Birr+200 Birr=2200 Birr.

Profit: $\text{Sale} - \text{Purchase} = 2200\text{Birr} - 1800\text{Birr} = 400\text{Birr}$.

Therefore, the profit of the man on that particular day is 400 Birr.

Example 4

Assume that for a closed economy, E = C + I + G, where E is total expenditure, C is expenditure on consumption of goods, I is expenditure on investment on goods and G is Government spending.

For equilibrium, we must have E = Y, where Y is the total income received.

For a certain economy, it is given that $C = 15 + 0.90\ Y$, $I = 20 + 0.05\ Y$ and $G = 25$.

a. Find the equilibrium values of Y, C and I.

b. How will these change if there is no Government spending?

Solution

Given that $E = C + I + G$ and $E = Y$. Thus, we have

a. $Y = C + I + G = (15 + 0.90\ Y) + (20 + 0.05\ Y) + 25 = 60 + 0.95\ Y$

This implies $Y(1 - 0.95) = 60 \Rightarrow Y = \dfrac{60}{0.05} = 1200$

For this value of Y, we have $C = 15 + 0.90\ Y = 15 + 0.90 \times 1200 = 1095$

and $I = 20 + 0.05\ Y = 20 + 0.05 \times 1200 = 80$

b. If there is no government spending, that is, $G = 0$, then closed economy equation becomes $Y = C + I = (15 + 0.90\ Y) + (20 + 0.05\ Y) = 35 + 0.95\ Y$

or $Y(1 - 0.95) = 35$, that is, $Y = \dfrac{35}{0.05} = 700$

For this value of Y, we have, $C = 15 + 0.90\ Y = 15 + 0.90 \times 700 = 645$

and $I = 20 + 0.05\ Y = 20 + 0.05 \times 700 = 55$.

Exercise 1.24

1. An electrician charges a base fee of Birr70 plus Birr 150 for each hour of work.
   a. Create a table that shows the amount that the electrician charges for 1,2, 3, and 4 hours of work.
   b. Let x represent the number of hours and y represent the amount charged for x hours. Is this relation a function?

2. A firm produces an item whose production cost function is $C(x)=200+60x$, where x is the number of items produced.
   a. If entire stock is sold at a price of each item which is Birr 800, then determine the revenue function.
   b. If the total number of items produced was 1000 and entire stock is sold at a price of each item which is Birr 800, find the profit of the firm.

Problem Solving

1. A company producing dry cells introduces production bonus for its employees which increases the cost of production. The daily cost of production $C(x)$ for x number of cells is Birr $(3.5x + 12{,}000)$.
   a. If each cell is sold for Birr 6, determine the number of cells that should be produced to ensure no loss.
   b. If the selling price is increased by 50 percent, what would be the break-even point?
   c. If at least 6000 cells can be sold daily, what price the company should charge per cell to guarantee no loss?
2. The force applied to a spring varies directly with the distance that the spring is stretched. When 30 N of force is applied, the spring stretches 3 m.
   a. Write a variation model using k as the constant of variation.
   b. Find k.
   c. How many meters will the spring stretch when 5N force pressure is applied?

Summary

1. A relation is any set of ordered pairs.
2. Given two sets A and B, a relation from A to B is any subset of $A \times B$.
3. Given a relation R, the inverse of R is given by $R^{-1} = \{(y,x): (x,y) \in R\}$
4. If R is a relation, then Domain of R^{-1} = Range of R and Range of R^{-1} = Domain of R.
5. The graph of the inverse of a given relation can be obtained by reflecting the graph of the relation with respect to the line $y = x$.
6. A function is a special type of a relation in which no two of the ordered pairs in it have the same first element.
7. A function of the form $f(x)=ax^r$, where a is a nonzero real number and r is a real number is called power function.
8. The function defined by $f(x)=|x|$, where $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ is called the absolute value function.
9. The function defined by $f(x)=\operatorname{sgn} x$, where $\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$ is called the signum function.
10. The function defined by $f(x)=x$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x is called the floor function or the greatest integer function.
11. If f is a function from A into B and g is a function from B into C, then the composition of f by g is the function defined by $(f \circ g)(x)=f(g(x))$.
12. For two functions f and g, g and f are inverse functions of each other if and only if $g(f(x))=x$ and $f(g(x))=x$.

Review Exercise

1. Find the domain and range of each of the following relations.

   a. $R_1 = \{(1,0),(2,2),(3,2),(4,5)\}$.

   b. $R_2 = \{(-1,-2),(0,1),(1,2),(2,5)\}$.

   c. $R_3 = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } y = x - 1\}$.

   d. $R_4 = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } y = x^2 + 1\}$.

2. Find inverse, domain and range of the inverses of each of the relations in Question Number 1 above.

3. Find the inverse of each of the following functions.

   a. $f(x) = 3x - 2$ b. $g(x) = x^3 + 5$ c. $h(x) = \dfrac{2x+3}{4x-5}$

4. Find the domain and range of each of the following functions.

   a. $f(x) = 3x^{\frac{4}{5}}$ b. $g(x) = -2x^{\frac{5}{4}}$ c. $h(x) = x^{\frac{3}{5}}$

5. Given $f(x) = 3 + 2|x|$:

   a. evaluate f at each of the following numbers.

      i. -5 ii. 0 iii. $-\frac{5}{2}$ iv. 3

   b. find the domain and range of f.

   c. sketch the graph of f.

6. Given $f(x) = 2 - 3sgn\ x$:

   a. evaluate f at each of the following numbers.

      i. -3 ii. 0 iii. $\frac{1}{2}$ iv. 4

   b. find the domain and range of f.

   c. sketch the graph of f.

7. Let $f(x) = \lfloor x - 3 \rfloor$. Then
   a. evaluate f at each of the following numbers.
      i. -2 ii. 0 iii. $-\frac{5}{4}$ iv. 5
   b. find the domain and rang of f.
   c. sketch the graph of f.
8. Determine if each pair of the following functions are inverses of each other.
   a. $f(x) = x - 1$ and $g(x) = x + 1$
   b. $f(x) = \frac{2x+3}{x-1}$ and $g(x) = \frac{x+3}{2x+1}$
   c. $f(x) = 2x - 3$ and $g(x) = 3 - 2x$
9. The area A of a square is directly proportional to the square of the length s of its sides.
   a. Write a general model of the proportionality with k as the constant of proportion.
   b. If the length of the sides is doubled, what effect will that have on the area?
   c. If the length of the sides is tripled, what effect will that have on the area?

UNIT 2

RATIONAL EXPRESSIONS AND RATIONAL FUNCTIONS

Unit Outcomes

By the end of this unit, you will be able to:

- Recognize a rational expression.
- Define a rational expression and a rational function.
- Know methods and procedures in simplifying rational expressions.
- Understand efficient methods in solving rational equations and inequalities.
- Develop efficient methods in solving rational equations and inequalities.
- Know basic concept and specific facts about rational functions.

Unit Contents

KEY Words

- graphs of rational functions
- horizontal asymptote
- oblique asymptote
- operations on rational expressions
- partial fractions
- rational equations
- rational expression
- rational inequality
- vertical asymptote
- zeros of a rational function

Introduction

You can add, subtract or multiply polynomial expressions using the basic rules defined on the set of real numbers. Then, in these computational processes the result gives another polynomial expression. On the other hand, if you divide a polynomial expression by another polynomial expression, the result may not be a polynomial expression. In this unit, you will learn about quotients of polynomial expressions which are called rational expressions.

2.1 Rational Expressions

2.1.1 Rational Expressions

Activity 2.1

1. Which of the following numbers are rational numbers?

   a. $1.\overline{12}$ b. $\frac{2}{3}$ c. π d. $\sqrt{2}$

2. Which of the following are polynomial expressions?

   a. $x^3 + 2x - 1$ b. $3\sqrt{x} + 5x + 3$ c. $3^x + x - 9$

From your responses in Activity 2.1, observe that:

a. a rational number is a number that can be written as a quotient of two integers.

b. An expression that can be written in the form $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where n is a non-negative integer and $a_0, \ldots, a_n$ are real numbers, is called a polynomial expression.

Rational expressions are defined as quotients of polynomials.

Definition 2.1

Given two polynomials $p(x)$ and $q(x)$ with $q(x) \neq 0$, an expression that can be expressed in the form $\frac{p(x)}{q(x)}$ is called a rational expression. In a rational expression $\frac{p(x)}{q(x)}$, the polynomial $p(x)$ is called numerator and $q(x)$ is called denominator.

Example

Which of the following are rational expressions?

a. $\frac{2x+3}{5x^2+9x-4}$ b. $\frac{6x^3+4x-7}{x^2+1}$ c. $\frac{2}{x^3-1}$

d. x^3+3x-1 e. $\frac{2x^2+x}{\sqrt{x^2+1}}$

Solution

a. $\frac{2x+3}{5x^2+9x-4}$ is a rational expression, because both $2x+3$ and $5x^2+9x-4$ are polynomial expressions.

b. $\frac{6x^3+4x-7}{x^2+1}$ is a rational expression, because both $6x^3+4x-7$ and x^2+1 are polynomial expressions.

c. $\frac{2}{x^3-1}$ is a rational expression, because 2 is a constant polynomial and x^3-1 is a polynomial expression.

d. x^3+3x-1 is a rational expression, because the expression x^3+3x-1 can be expressed as $\frac{x^3+3x-1}{1}$ and both x^3+3x-1 and 1 are polynomial expressions.

e. $\frac{2x^2+x}{\sqrt{x^2+1}}$ is not a rational expression, because the denominator $\sqrt{x^2+1}$ is not a polynomial expression.

Exercise 2.1

Which of the following expressions are rational expressions?

a. $\frac{x+3}{x^3+x-9}$

b. $\frac{1}{x^3+\sqrt{x}+1}$

c. $\frac{x^2-x}{\sqrt{x^4+4x^2+4}}$

d. $\frac{x^{\frac{3}{2}}+2x-1}{6x^3+4x-7}$

e. $2x^4-3x^3+2x^2+3x-1$

2.1.2 Domains of Rational Expressions

Activity 2.2

Determine the value of the rational expression $\frac{2x^2+x}{x^2-1}$ for each of the following values of x if it is defined.

a. $x=2$ b. $x=-1$ c. $x=1$ d. $x=0$

From your responses in Activity 2.2, observe that a given rational expression is not defined for those numbers that are the zeros of the expression x^2-1, which is the denominator of the given rational expression.

Definition 2.2

Given a rational expression $\frac{p(x)}{q(x)}$, the set of all real numbers x such that $q(x)\neq 0$ is called the domain of $\frac{p(x)}{q(x)}$. That is, the domain of $\frac{p(x)}{q(x)}$ is $\{x\in\mathbb{R}: q(x)\neq 0\}$.

Example

Determine the domain of each of the following rational expressions:

a. $\frac{2x^2+x}{6x+3}$ b. $\frac{3}{x^2+5x+6}$ c. $\frac{5x+9}{x^2+1}$

Solution

a. The denominator of the given rational expression is $6x+3$ and $6x+3=0 \Rightarrow x=-\frac{1}{2}$.

Therefore, the domain of $\frac{2x^2+x}{6x+3}$ is $\left\{x \in \mathbb{R} \mid x \neq -\frac{1}{2}\right\}$.

b. The denominator of the given rational expression is x^2+5x+6 and $x^2+5x+6=0$

implies $(x+2)(x+3)=0 \Rightarrow x=-2$ or $x=-3$.

Thus, the domain of $\frac{3}{x^2+5x+6}$ is $\{x \in \mathbb{R} \mid x \neq -2 \text{ and } x \neq -3\}$.

c. The denominator of the given rational expression x^2+1 and $x^2+1 \neq 0$ for all $x \in \mathbb{R}$. Therefore, domain of $\frac{5x+9}{x^2+1}$ is $\mathbb{R}$, the set of all real numbers.

2.1.3 Simplifications of Rational Expressions

Activity 2.3

Write each one of the following rational numbers in its lowest term.

a. $\frac{4}{6}$ b. $\frac{10}{100}$ c. $\frac{15}{28}$ d. $\frac{18}{3}$

From your responses in Activity 2.3, observe the following two important points to write a given rational number in its lowest form:

i. first factorize both the numerator and the denominator; and

ii. cancel out the common factors of both the numerator and the denominator.

Recall from your knowledge about rational numbers that:

i. a rational number $\frac{a}{b}$, where a and b are integers and $b \neq 0$, is in its lowest term if the greatest common factor of a and b is 1;

ii. given three integers a, b, c, with $b \neq 0$ and $c \neq 0$, you have:

$$\frac{ac}{bc} = \frac{a}{b}.$$

Definition 2.3

A rational expression $\frac{p(x)}{q(x)}$ is said to be in lowest term or in its simplest form, if only common factor of the numerator $p(x)$ and the denominator $q(x)$ is 1.

Example 1

Simplify the rational expression $\frac{x+1}{x^2-1}$.

Solution

$\frac{x+1}{x^2-1} = \frac{(x+1)\times 1}{(x+1)(x-1)}$ (Factorize both the numerator and the denominator.)

$= \frac{1}{x-1}$, for $x \neq -1$. (Cancel out the common factor, $x+1$, of both the numerator and denominator.)

Observe that for $x = -1$, the expression $\frac{x+1}{x^2-1}$ is not defined, but the expression $\frac{1}{x-1}$ is defined at $x = -1$, and its value is $\frac{1}{-1-1} = -\frac{1}{2}$.

Therefore, $\frac{x+1}{x^2-1} = \frac{1}{x-1}$, for $x \neq -1$.

To simplify a given rational expression, first you have to determine the domain of the given expression.

Steps to simplify a given rational expression $\frac{p(x)}{q(x)}$:

Step 1: Determine the domain of the expression $\frac{p(x)}{q(x)}$;

Step 2: Factorize both $p(x)$ and $q(x)$ (if possible); and

Step 3: Cancel out the common factors of both $p(x)$ and $q(x)$ to obtain the simplified form of the given rational expression.

Example 2

Simplify each of the following rational expressions.

a. $\frac{3x+6}{x^2+4x+4}$ b. $\frac{x^3+2x^2+x}{x^2-1}$ c. $\frac{x-1}{x^2-4x+3}$

Solution

a. $x^2+4x+4=0$ implies $x=-2$. Then, the domain of $\frac{3x+6}{x^2+4x+4}$ is $\{x \in \mathbb{R} | x \neq -2\}$.

Then $\frac{3x+6}{x^2+4x+4} = \frac{3(x+2)}{(x+2)^2}$ (Factorize both $3x+6$ and x^2+4x+4.)

$= \frac{3(x+2)}{(x+2)(x+2)} = \frac{3}{x+2}$, for $x \neq -2$ (Cancel out $x+2$ from both the numerator and the denominator.)

Therefore, $\frac{3x+6}{x^2+4x+4} = \frac{3}{x+2}$, for $x \neq -2$ in a simplified form as 1 is the only common factor of 3 and $x+2$.

b. $x^2-1=0$ implies $x=1$ or $x=-1$.

Then domain of $\frac{x^3+2x^2+x}{x^2-1}$ is $\{x \in \mathbb{R} | x \neq -1, 1\}$.

$\frac{x^3+2x^2+x}{x^2-1} = \frac{x(x+1)(x+1)}{(x+1)(x-1)}$ (Factorize both $x^3 + 2x^2 + x$ and $x^2 - 1$.)

$= \frac{x(x+1)}{x-1} = \frac{x^2+x}{x-1}$, for $x \neq -1$ (Cancel out $x + 1$ from the numerator and the denominator.)

Therefore, $\frac{x^3+2x^2+x}{x^2-1} = \frac{x^2+x}{x-1}$ for $x \neq -1$.

c. $x^2 - 4x + 3 = 0$ implies $x = 1$ and $x = 3$.

Then the domain of $\frac{x-1}{x^2-4x+3}$ is $\{x \in \mathbb{R} | x \neq 1, 3\}$.

$\frac{x-1}{x^2-4x+3} = \frac{(x-1)}{(x-1)(x-3)}$ (Factorize $x^2 - 4x + 3$.)

$= \frac{(x-1)\times 1}{(x-1)(x-3)} = \frac{1}{x-3}$, for $x \neq -1$ (Cancel out $x - 1$ from both the numerator and the denominator.)

Therefore, $\frac{x-1}{x^2-4x+3} = \frac{1}{x-3}$ for $x \neq 1$.

Exercise 2.2

1. Find the domain of each of the following rational expressions.

a. $\frac{2}{x^2+3}$ b. $\frac{x+1}{x^2-4}$ c. $\frac{x^4-5x+2}{x^3+6x^2+5x}$

2. Simplify each of the following rational expressions.

a. $\frac{2x-4}{x^2-x-2}$ b. $\frac{x^2+5x+6}{x^2-9}$ c. $\frac{x+2}{x^2+2x+2}$

2.1.4 Operations with Rational Expressions

Revision of Operations on Rational Numbers

Activity 2.4

Compute each of the following.

a. $\frac{1}{2}+\frac{2}{3}$ b. $\frac{1}{2}-\frac{2}{3}$ c. $\frac{1}{2}\times\frac{2}{3}$ d. $\frac{1}{2}\div\frac{2}{3}$

From your knowledge on rational numbers, observe that the results in Activity 2.4 are obtained by using the rules of addition, subtraction, multiplication and division of rational numbers.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be rational numbers with $b \neq 0$ and $d \neq 0$. The basic four operations, addition, subtraction, multiplication and division on rational numbers are given as follows:

i. $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$

ii. $\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$

iii. $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$

iv. $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$, $(c \neq 0)$

Addition and Subtraction of Rational Expressions

Addition and subtraction of rational expressions are extensions of the same operations on the set of rational numbers and they are defined as follows:

consider two rational expressions $\frac{p(x)}{q(x)}$ and $\frac{r(x)}{s(x)}$, with $q(x) \neq 0$ and $s(x) \neq 0$. Then,

a. $\frac{p(x)}{q(x)}+\frac{r(x)}{s(x)}=\frac{p(x)s(x)+r(x)q(x)}{q(x)s(x)}$

b. $\frac{p(x)}{q(x)}-\frac{r(x)}{s(x)}=\frac{p(x)s(x)-r(x)q(x)}{q(x)s(x)}$

Example 1

Perform each of the following operations:

a. $\frac{x+1}{x^2+2x}+\frac{3x}{x+5}$

b. $\frac{x+1}{x^2+2x}-\frac{3x}{x+5}$

Solution

a. $$\frac{x+1}{x^2+2x}+\frac{3x}{x+5}=\frac{(x+1)(x+5)+3x(x^2+2x)}{(x^2+2x)(x+5)}=\frac{x^2+6x+5+3x^3+6x^2}{x^3+7x^2+10x}$$

Therefore, $\frac{x+1}{x^2+2x}+\frac{3x}{x+5}=\frac{3x^3+7x^2+6x+5}{x^3+7x^2+10x}$.

b. $\frac{x+1}{x^2+2x}-\frac{3x}{x+5}=\frac{(x+1)(x+5)-3x(x^2+2x)}{(x^2+2x)(x+5)}=\frac{x^2+6x+5-3x^3-6x^2}{x^3+7x^2+10x}$

Therefore, $\frac{x+1}{x^2+2x}-\frac{3x}{x+5}=\frac{-3x^3-5x^2+6x+5}{x^3+7x^2+10x}$.

Multiplication and Division of Rational Expressions

Multiplication and division of rational expressions are extensions of the same operations on the set of rational numbers and they are defined as follows:

consider two rational expressions $\frac{p(x)}{q(x)}$ and $\frac{r(x)}{s(x)}$, with $q(x)\neq 0$ and $s(x)\neq 0$. Then,

i. $\frac{p(x)}{q(x)}\times\frac{r(x)}{s(x)}=\frac{p(x)r(x)}{q(x)s(x)}$;

ii. $\frac{p(x)}{q(x)}\div\frac{r(x)}{s(x)}=\frac{p(x)}{q(x)}\times\frac{s(x)}{r(x)}=\frac{p(x)s(x)}{q(x)r(x)}$, $(r(x)\neq 0)$.

Example 2

Perform each of the following operations:

a. $\frac{x+1}{x-1}\times\frac{x+2}{x^2+3x}$

b. $\frac{x+1}{x-1}\div\frac{x+2}{x^2+3x}$

Solution

a. $\frac{x+1}{x-1}\times\frac{x+2}{x^2+3x}=\frac{(x+1)(x+2)}{(x-1)(x^2+3x)}=\frac{x^2+3x+2}{x^3+2x^2-3x}$

b. $\frac{x+1}{x-1}\div\frac{x+2}{x^2+3x}=\frac{x+1}{x-1}\times\frac{x^2+3x}{x+2}=\frac{(x+1)(x^2+3x)}{(x-1)(x+2)}=\frac{x^3+4x^2+3x}{x^2+x-2}$

Exercise 2.3

Perform each of the following operations and simplify the given expression.

a. $\frac{x-1}{x+2}+\frac{x+3}{x-4}$ b. $\frac{x-1}{x+2}-\frac{x+3}{x-4}$

c. $\frac{x-1}{x+2}\times\frac{x+3}{x-4}$ d. $\frac{x-1}{x+2}\div\frac{x+3}{x-4}$

2.1.5 Decomposition of Rational Expressions into Partial Fractions

Activity 2.5

Consider the rational expression $\frac{x+1}{x^2-3x+2}$. Then,

a. factorize x^2-3x+2; b. show that $\frac{x+1}{x^2-3x+2}=\frac{-2}{x-1}+\frac{3}{x-2}$.

In Activity 2.5, the fractional expressions $\frac{-2}{x-1}$ and $\frac{3}{x-2}$ are called partial fractions and writing $\frac{x+1}{x^2-3x+2}$ as $\frac{-2}{x-1}+\frac{3}{x-2}$ is decomposing the given expression as a sum of partial fractions.

The main objective in this discussion is to determine how to write a given rational expression as a sum of partial fractions. The idea of writing a rational expression as a sum of partial fractions is important, especially in a branch of Mathematics called Calculus and you will be introduced to the topic calculus in Grade 12.

Definition 2.4

Two polynomials $p(x)$ and $q(x)$ are equal if both polynomials have the same degree and the terms of the same degree in both polynomials have equal coefficients.

Example 1

Let $p(x)=3x^3+5x^2-7x+c$ and $q(x)=ax^4+bx^3+5x^2-7x+4$.

Find the values of the constants a, b and c if $p(x)=q(x)$ for all $x\in\mathbb{R}$.

Solution

$p(x) = q(x)$ for all $x \in \mathbb{R}$ if and only if terms of the same degree in both polynomials have equal coefficients.

i. The coefficient of the term with degree 4 in $p(x)$ is 0 and the coefficient of the term with degree 4 in $q(x)$ is a. This implies, $a = 0$.

ii. The coefficient of the term with degree 3 in $p(x)$ is 3 and the coefficient of the term with degree 3 in $q(x)$ is b. This implies, $b = 3$.

iii. The constant term in $p(x)$ is c and the constant term in $q(x)$ is 4. This implies, $c = 4$.

Definition 2.5

A rational expression $\frac{p(x)}{q(x)}$ is called proper rational expression if the degree of $p(x)$ is less than degree of $q(x)$, otherwise it is called an improper rational expression.

Note

Note that, given an improper rational expression $\frac{p(x)}{q(x)}$, using long division of polynomial expressions, we can write it as $\frac{p(x)}{q(x)} = f(x) + \frac{r(x)}{q(x)}$, where $f(x)$ and $r(x)$ are polynomial expressions and the degree of $r(x)$ is less than the degree of $q(x)$.

Example 2

Write the rational expression $\frac{5x^3+7x^2+5x+9}{x^2+1}$ in the form of $f(x) + \frac{r(x)}{q(x)}$, where $f(x)$ and $r(x)$ are polynomial expressions and the degree of $r(x)$ is less than the degree of $q(x)$.

Solution

Using long division on polynomial expressions, the polynomial expression $5x^3 + 7x^2 + 5x + 9$ can be written as $5x^3 + 7x^2 + 5x + 9 = (x^2 + 1)(5x + 7) + 2$.

Then

$$\frac{5x^3 + 7x^2 + 5x + 9}{x^2 + 1} = (5x + 7) + \frac{2}{x^2 + 1}$$

In writing a given rational expression as a sum of partial fractions, it is always important to factorize a polynomial expression into a product of polynomial expressions of smaller degree whenever possible.

The following Theorem is important in factorizing a polynomial into a product of polynomials of smaller degrees.

Theorem 2.1

Any non-constant polynomial expression with real coefficients can be factorized as a product of linear and/or irreducible quadratic factors with the possibility of some powers.

Example 3

Factorize each of the following polynomials, if possible.

a. $x^2 + 5x + 6$

b. $x^3 - 4x^2 + 7x - 6$

Solution

a. $x^2 + 5x + 6 = (x + 2)(x + 3)$

b. $x^3 - 4x^2 + 7x - 6 = (x - 2)(x^2 - 2x + 3)$ and the quadratic polynomial $x^2 - 2x + 3$ is irreducible as its discriminant $(-2)^2 - 4(1)(3) = -8$ and $-8 < 0$.

Exercise 2.4

1. Determine the constants a, b, c and d, if the polynomials $p(x) = ax^5 + 5x^4 + 3x^2 + d$ and $q(x) = 2x^5 + bx^4 + cx^3 + 3x^2 - 4$ are equal for all $x \in \mathbb{R}$.
2. Write each of the following rational expressions as $f(x) + \frac{r(x)}{q(x)}$, where $f(x), r(x)$ and $q(x)$ are polynomials and the degree of $r(x)$ is less than the degree of $q(x)$.

   a. $\dfrac{x^3+1}{x^2+2x+5}$ b. $\dfrac{x^2-5}{x^2+1}$ c. $\dfrac{x+1}{x^3+4x+5}$

3. Factorize each of the following polynomials, if possible.

   a. $x^3 + x^2 - 2x$ b. $x^4 + 4x^2 + 4$

Steps in decomposing a given rational expression as a sum of partial fractions:

I. If $ax + b$, for $a \neq 0$, is a factor in the denominator of the given rational expression and $(ax + b)^2$ is not a factor, then the partial fraction that corresponds to this factor is $\frac{A}{ax+b}$, where A is a constant that we have to determine.

Example 4

Decompose $\frac{6x+2}{x^2+6x+5}$ as a sum of partial fractions.

Solution

The given rational expression is a proper rational expression because the degree of $6x + 2$ is 1, which is less than the degree of $x^2 + 6x + 5$, which is 2.

First let us factorize $x^2 + 6x + 5$.

That is, $x^2 + 6x + 5 = (x + 5)(x + 1)$.

Then, $\frac{6x+2}{x^2+6x+5} = \frac{A}{x+5} + \frac{B}{x+1}$ for some constants A and B.

Now, let us determine A and B and to do so write:

$$\frac{6x+2}{x^2+6x+5} = \frac{A}{x+1} + \frac{B}{x+5} = \frac{A(x+5)+B(x+1)}{(x+1)(x+5)} = \frac{(A+B)x+(5A+B)}{x^2+6x+5}$$

Then, for $x \neq -1$ and $x \neq -5$, the polynomial expressions, $6x + 2$ and $(A + B)x + (5A + B)$ are equal.

That is, $A + B = 6$ and $5A + B = 2$, the coefficients of terms of the same degree are equal.

Then we can solve the two equations simultaneously as follows:

$$-\begin{cases} A + B = 6 \\ 5A + B = 2 \end{cases}$$
$$\overline{\quad -4A = 4 \quad}$$

This implies $A = -1$ and from the equation $A + B = 6$, we have $B = 6 - (-1) = 7$.

Hence, $\frac{6x+2}{x^2+6x+5} = \frac{-1}{x+1} + \frac{7}{x+5}$ (written as a sum of partial fractions).

II. If $(ax+b)^k$, for $a \neq 0$, is a factor in the denominator, for some integer $k > 1$ and $(ax+b)^{k+1}$ is not a factor, then the corresponding partial fractions that correspond to this factor are $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$, where $A_1, \cdots, A_k$ are constants that we have to determine.

Example 5

Decompose $\frac{x+5}{x^2+4x+4}$ as a sum of partial fractions.

Solution

First let us factorize x^2+4x+4.

That is, $x^2+4x+4=(x+2)^2$.

Then, $\frac{x+5}{x^2+4x+4}=\frac{A_1}{x+2}+\frac{A_2}{(x+2)^2}=\frac{A_1(x+2)+A_2}{(x+2)^2}=\frac{A_1x+(2A_1+A_2)}{(x+2)^2}$.

For all $x \in \mathbb{R}$ and $x \neq -2$, we have $x+5=A_1x+(2A_1+A_2)$.

This implies $A_1=1$ and $2A_1+A_2=5$ (Coefficients of terms of the same degree are equal). Then, solving for A_2 gives you $A_2=5-2A_1=5-2(1)=3$.

Therefore, $\frac{x+5}{x^2+4x+4}=\frac{1}{x+2}+\frac{3}{(x+2)^2}$ (written as a sum of partial fractions).

Exercise 2.5

Decompose each of the following rational expressions as a sum of partial fractions.

a. $\frac{x+1}{x^2+4x+3}$

b. $\frac{2x-3}{x^2+x-2}$

c. $\frac{x+2}{x^2+6x+9}$

d. $\frac{x-1}{x^2-4x+4}$

III. If ax^2+bx+c, for $a \neq 0$, is a factor in the denominator with $b^2-4ac<0$ and $(ax^2+bx+c)^2$ is not a factor, then the partial fraction that correspond to this factor is $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants that we have to determine.

Example 6

Write $\frac{x^2+3x+5}{(x+2)(x^2+x+1)}$ as a sum of partial fractions.

Solution

The quadratic expression x^2+x+1 is irreducible, because the discriminant:

$$1^2-4(1)(1)=-3 \text{ and } -3<0.$$

Then, $\frac{x^2+3x+5}{(x+2)(x^2+x+1)}=\frac{A}{x+2}+\frac{A_1x+B_1}{x^2+x+1}$.

This implies,

$$\frac{x^2+3x+5}{(x+2)(x^2+x+1)} = \frac{A\left(x^2+x+1\right)+(A_1x+B_1)(x+2)}{(x+2)(x^2+x+1)}$$

$$= \frac{\left(A+A_1\right)x^2+\left(A+2A_1+B_1\right)x+(A+2B_1)}{(x+2)(x^2+x+1)}.$$

Thus, for $x\neq-2$, $x^2+3x+5=(A+A_1)x^2+\left(A+2A_1+B_1\right)x+(A+2B_1)$.

Thus, from the equality of the two polynomial expressions, we have

$$\begin{cases} A+A_1=1 \\ A+2A_1+B_1=3 \\ A+2B_1=5 \end{cases}$$

Solving the three equations simultaneously gives us $A=1$, $A_1=0$ and $B_1=2$.

Therefore, $\frac{x^2+3x+5}{(x+2)(x^2+x+1)}=\frac{1}{x+2}+\frac{2}{x^2+x+1}$ (written as a sum of partial fractions).

IV. If $(ax^2+bx+c)^k$, for $a \neq 0$ and $b^2-4ac<0$, is a factor in the denominator for $k>1$ and $\left(ax^2+bx+c\right)^{k+1}$ is not a factor, then the corresponding sum of partial fractions to this factor is:

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{\left(ax^2+bx+c\right)^k},$$

where $A_1,\ldots,A_k$ and $B_1,\ldots,B_k$ are constants that we have to determine.

Example 7

Write $\frac{6x^2+5x+2}{(x^2+2x+3)^2}$ as a sum of partial fractions.

Solution

The quadratic expression x^2+2x+3 is irreducible, because

$2^2-4(1)(3)=4-12=-8$ and $-8<0$

Then, $\dfrac{6x^2+5x+2}{(x^2+2x+3)^2}=\dfrac{A_1x+B_1}{x^2+2x+3}+\dfrac{A_2x+B_2}{(x^2+2x+3)^2}$.

Thus

$6x^2+5x+2=A_1x^3+\left(2A_1+B_1\right)x^2+\left(3A_1+2B_1+A_2\right)x+(3B_1+B_2)$ for all $x \in \mathbb{R}$.

This implies, $A_1=0$, $2A_1+B_1=6, 3A_1+2B_1+A_2=5, 3B_1+B_2=2$ (the coefficients of terms of the same degree are equal).

Then, solving four equations simultaneously gives you,

$A_1=0,\ B_1=6,\ A_2=-7,\ B_2=-16$.

Therefore, $\dfrac{6x^2+5x+2}{(x^2+2x+3)^2}=\dfrac{6}{x^2+2x+3}+\dfrac{-7x-16}{(x^2+2x+3)^2}$ (written as a sum of partial fractions).

Note

To decompose an improper fraction of the form $\frac{p(x)}{q(x)}$, first write $\frac{p(x)}{q(x)} = f(x) + \frac{r(x)}{q(x)}$, where the degree of $r(x)$ is less than the degree of $q(x)$ and then write $\frac{r(x)}{q(x)}$ as a sum of partial fractions using any one of the methods discussed above.

Example 8

Write $\frac{x^3+7x+1}{x^2+3x+2}$ as a sum of partial fraction.

Solution

Since the degree of the numerator is not less than degree of the denominator, perform polynomial long division.

$$\frac{x^3+7x+1}{x^2+3x+2} = x-3+\frac{14x+7}{x^2+3x+2}$$

Then write $\frac{14x+7}{x^2+3x+2}$ as a sum of partial fractions.

That is, $\frac{14x+7}{x^2+3x+2} = \frac{14x+7}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$.

This implies $\frac{14x+7}{(x+1)(x+2)} = \frac{(x+1)B+(x+2)A}{(x+1)(x+2)} = \frac{(A+B)x+(2A+B)}{(x+1)(x+2)}$.

The denominators of the two expressions are equal, so we require the equality of the numerators:

$14x+7 = (A+B)x + (2A+B)$.

Thus, solve the following system of two linear equations:

$$\begin{cases} A+B=14 \\ 2A+B=7 \end{cases}$$

Solving these two equations gives you: $A=-7$ and $B=21$.

Therefore, $\frac{x^3+7x+1}{x^2+3x+2}=x-3+\frac{14x+7}{x^2+3x+2}=x-3+\frac{-7}{x+1}+\frac{21}{x+2}$.

Exercise 2.6

Write each of the following rational expressions as a sum of partial fractions.

a. $\frac{2x^2+6x-6}{x^3-1}$ b. $\frac{x^2+7x}{x^2+7x-8}$ c. $\frac{x^2-3}{x^3+x^2+x+1}$

d. $\frac{42-14x}{x^3-4x^2+x-4}$ e. $\frac{x^3+2x+1}{x^2+x-2}$

2.2 Rational Equations and Rational Inequalities

2.2.1 Rational Equations

Activity 2.6

1. Solve each of the following equations.

   a. $\frac{x+1}{5}=\frac{3}{2}$ b. $\frac{2x}{3}=\frac{1}{3}$

2. Solve each of the following equations;

   a. $x^2-4x+3=0$ b. $x^2-3x=0$

In your responses of Activity 2.6, you need to revise how to solve linear equations for question number 1 and quadratic equations for question number 2 from your earlier grade mathematics.

In this section, you will learn how to solve equations involving rational expressions.

Definition 2.6

Any equation that involves only rational expressions is called a rational equation. That is, a rational equation is an equation that can be reduced to the form $\frac{p(x)}{q(x)}=0$, where $p(x)$ and $q(x)$ are polynomial expressions and $q(x)\neq 0$.

Example 1

Write each of the following equations as $\frac{p(x)}{q(x)}=0$, where $p(x)$ and $q(x)$ are polynomial expressions and $q(x)\neq 0$.

a. $\frac{1}{x+1}=\frac{2x}{x-1}$ b. $3x=\frac{2x+5}{x^2-2}$

Solution

a. $\frac{1}{x+1}=\frac{2x}{x-1}$ implies $\frac{1}{x+1}-\frac{2x}{x-1}=0$.

$\Rightarrow \frac{1(x-1)-2x(x+1)}{(x+1)(x-1)}=0$. (Simplifying the rational expression.)

That is, $\frac{-2x^2-x-1}{x^2-1}=0$ is the required rational equation.

b. $3x=\frac{2x+5}{x^2-2}$ implies $3x-\frac{2x+5}{x^2-2}=0$

$\Rightarrow \frac{3x\left(x^2-2\right)-1(2x+5)}{x^2-2}=0$ (Simplifying the rational expression)

Thus, $\frac{3x^3-8x-5}{x^2-2}=0$ is the required rational equation.

Steps in Solving Rational Equations

Given a rational equation, solving the rational equation is finding all the possible numbers in the domain of the given rational expression that satisfy the given rational equation.

The following are steps that you can follow to solve rational equations.

Step 1: Reduce the given rational equation to the form $\frac{p(x)}{q(x)}=0$.(without simplifying the given expression).

Step 2: Solve the equation $q(x)=0$.

Step 3: Solve the equation $p(x)=0$.

Therefore, the solution set is the set of all real numbers in Step 3 without the set of numbers obtained in Step 2.

Example 2

Solve each of the following rational equations:

a. $x-\frac{1}{2}=\frac{1}{3}x$ b. $2-\frac{x}{x+1}=\frac{3}{x+1}$ c. $\frac{2}{x^2-9}=\frac{x}{x+3}$

Solution

a. $x-\frac{1}{2}=\frac{1}{3}x$ implies $x-\frac{1}{3}x-\frac{1}{2}=0$. (Subtracting $\frac{1}{3}x$ from both sides.)

Then, $\frac{6x-2x-3}{3\times 2}=0$ (Simplifying the rational expression.).

This implies, $\frac{4x-3}{6}=0 \Rightarrow 4x-3=0$ (a linear equation)

Then, solving the linear equation $4x-3=0$ gives us $x=\frac{3}{4}$.

Therefore, the solution set of the given equation is $\left\{\frac{3}{4}\right\}$.

b. $2-\frac{x}{x+1}=\frac{3}{x+1}$ implies $2-\frac{x}{x+1}-\frac{3}{x+1}=0$. (Subtracting $\frac{3}{x+1}$ from both sides.)

This implies, $\frac{2(x+1)-(x+3)}{x+1}=0 \Rightarrow \frac{x-1}{x+1}=0$.

Then, determine the value(s) of x that make the denominator zero.

That is, $x+1=0 \Rightarrow x=-1$.

Thus, $x=-1$ will be excluded from the solution set of the given rational equation.

Then, solving the equation $x-1=0$ gives us $x=1$.

Therefore, the solution set of the given equation is $\{1\}$.

c. $\frac{2}{x^2-9}=\frac{x}{x+3}$ implies $\frac{2}{x^2-9}-\frac{x}{x+3}=0$. (Subtracting $\frac{x}{x+3}$ from both sides.)

Then, $\frac{2}{(x-3)(x+3)}-\frac{x}{x+3}=0$.

This implies $\frac{2-x(x-3)}{(x-3)(x+3)}=0 \Longrightarrow \frac{2-x^2+3x}{(x-3)(x+3)}=0$.

Find the value(s) of x that make the denominator zero.

That is, $(x-3)(x+3)=0 \Longrightarrow x-3=0$ or $x+3=0$

That is, $x=3$ or $x=-3$ and both $x=3$ and $x=-3$ are excluded from the solution set of the given rational equation.

Then, $2-x^2+3x=0$ and this is a quadratic equation. Thus, using the quadratic formula:

$$x=\frac{-3\pm\sqrt{9-4(-1)(2)}}{2(-1)}$$

That is, $x=\frac{-3\pm\sqrt{17}}{-2}$.

Therefore, the solution set of the given equation is $\left\{\frac{3-\sqrt{17}}{2},\frac{3+\sqrt{17}}{2}\right\}$.

Exercise 2.7

Solve each of the following equations.

a. $\dfrac{x^2-5x+6}{x^2-2}=0$

b. $\dfrac{x}{x-1}=\dfrac{3}{x+1}$

2.2.2 Rational Inequalities

Definition 2.7

Any inequality that involves only rational expressions is called a rational inequality.

That is, a rational inequality is an inequality that can be reduced to the form $\dfrac{p(x)}{q(x)}\leq 0$, $\dfrac{p(x)}{q(x)}<0$, $\dfrac{p(x)}{q(x)}\geq 0$ or $\dfrac{p(x)}{q(x)}>0$, where $p(x)$ and $q(x)$ are polynomial expressions and $q(x)\neq 0$.

Example 1

Which of the following are rational inequalities?

a. $1>\dfrac{2}{3x}$

b. $2^x<\dfrac{2}{x+5}$

c. $\dfrac{1}{x}-\dfrac{2}{x^2}\leq\dfrac{5}{x^2+1}$

Solution

a. Since both 1 and $\frac{2}{3x}$ are rational expressions, $1>\frac{2}{3x}$ is a rational inequality.

b. 2^x is not a rational expression. Thus, $2^x<\frac{2}{x+5}$ is not a rational inequality

c. $\frac{1}{x}-\frac{2}{x^2}\leq\frac{5}{x^2+1}\Leftrightarrow\frac{x-2}{x^2}\leq\frac{5}{x^2+1}$ and both $\frac{x-2}{x^2}$ and $\frac{5}{x^2+1}$ are rational expressions.

Thus, $\frac{1}{x}-\frac{2}{x^2}\leq\frac{5}{x^2+1}$ is a rational inequality.

The solution set of a rational inequality is the set of all real numbers in the domain of the given rational expression that satisfy the given inequality.

Steps in Solving Rational Inequalities

Step 1: Determine the domain of the rational inequality.

Step 2: Write the given inequality to the form $\frac{p(x)}{q(x)} > 0$, $\frac{p(x)}{q(x)} \geq 0$, $\frac{p(x)}{q(x)} < 0$ or $\frac{p(x)}{q(x)} \leq 0$.

Step 3: Find the zeros of both $p(x)$ and $q(x)$.

Step 4: Divide the number line into different intervals using the numbers that are not in the domain of the inequality in Step 1 and numbers in the Step 3.

The method is called the sign chart method.

Step 5: Find the interval that satisfies the given inequality using the product rule.

Example 2

Solve the inequality $\frac{x+1}{x-1} > 0$.

Solution

To determine the domain of the given inequality, solve the equation $x - 1 = 0$.

This implies, $x = 1$ and then the domain of the given inequality is $\{x \in \mathbb{R} | x \neq 1\}$.

Solve the equation $x + 1 = 0$. This implies $x = -1$.

Using the numbers -1 and 1, divide the number line into three intervals as in the table below.

	$x < -1$	-1	$-1 < x < 1$	1	$x > 1$
$x + 1$	$- - -$	0	$+ + +$	$+$	$+ + +$
$x - 1$	$- - -$	$-$	$- - -$	0	$+ + +$
$\frac{x+1}{x-1}$	$+ + +$	0	$- - -$	Undefined	$+ + +$

Then, from the given sign chart, observe that:

i. $\frac{x+1}{x-1}$ is positive on $(-\infty,-1) \cup (1,\infty)$;

ii. $\frac{x+1}{x-1}$ is negative on $(-1,1)$;

iii. $\frac{x+1}{x-1}$ is zero at $x=-1$ and

iv. $\frac{x+1}{x-1}$ is undefined at $x=1$.

Therefore, the solution set of the given inequality is $(-\infty,-1)\cup(1,\infty)$.

Example 3

Solve the inequality $\frac{1}{3}-\frac{2}{x^2}\geq\frac{5}{3x}$.

Solution

The given rational inequality is not defined when $x=0$.

Then, the domain of the given inequality is $\{x \in \mathbb{R} | x \neq 0\}$.

$\frac{1}{3}-\frac{2}{x^2}\geq\frac{5}{3x} \Rightarrow \frac{1}{3}-\frac{2}{x^2}-\frac{5}{3x}\geq 0$, which implies, $\frac{x^2-6-5x}{3x^2}\geq 0$.

Hence, the given inequality $\frac{1}{3}-\frac{2}{x^2}\geq\frac{5}{3x}$ is equivalent to the inequality $\frac{x^2-6-5x}{3x^2}\geq 0$, that is, both inequalities have the same solution set.

$$x^2-6-5x=0 \Rightarrow x^2-5x-6=0 \Rightarrow (x-6)(x+1)=0$$

This implies $x-6=0$ or $x+1=0 \Rightarrow x=6$ or $x=-1$.

Then, divide the number line into four different intervals using the numbers -1, 0 and 6 as in the table below.

	$x<-1$	-1	$-1<x<0$	0	$0<x<6$	6	$x>6$
$x-6$	$---$	$-$	$---$	$-$	$---$	0	$+++$
$x+1$	$---$	0	$+++$	$+$	$+++$	$+$	$+++$
$3x^2$	$+++$	$+$	$+++$	0	$+++$	$+$	$+++$
$\frac{(x-6)(x+1)}{3x^2}$	$+++$	0	$---$	Undefined	$---$	0	$+++$

Then, from the given sign chart, observe that:

i. $\frac{(x-6)(x+1)}{3x^2}$ is positive on $(-\infty,-1) \cup (6,\infty)$;

ii. $\frac{(x-6)(x+1)}{3x^2}$ is negative on $(-1,0) \cup (0,6)$;

iii. $\frac{(x-6)(x+1)}{3x^2}$ is zero at $x = -1$ and $x = 6$;

iv. $\frac{(x-6)(x+1)}{3x^2}$ is undefined at $x = 0$.

Therefore, the solution set of the inequality $\frac{x^2-6-5x}{3x^2} \geq 0$ is $(-\infty,-1] \cup [6,\infty)$, which is also the solution set of the inequality $\frac{1}{3} - \frac{2}{x^2} \geq \frac{5}{3x}$.

Exercise 2.8

Solve each of the following inequalities.

a. $\frac{x-1}{x+1} \geq 0$

b. $\frac{1}{3} + \frac{2}{x^2} < \frac{5}{3x}$

2.3 Rational Functions and Their Graphs

2.3.1 Rational Functions

Activity 2.7

Which of the following expressions are rational expressions?

a. $\frac{x+1}{x-1}$

b. $\frac{2}{x^2+3x-1}$

c. $\frac{\sqrt{x+1}}{x+2}$

From your responses in Activity 2.7, observe that rational expressions are expressions that can be written as a quotient of two polynomial expressions and that are used to define rational functions.

Definition 2.8

A function f that can be expressed in the form $f(x) = \frac{p(x)}{q(x)}$, where both $p(x)$ and $q(x)$ are polynomial expressions and $q(x) \neq 0$, is called a rational function.

Example 1

Which of the following are rational functions?

a. $f(x) = \frac{x^3+6x+9}{x^2+4x+1}$ b. $g(x) = \frac{3x-5}{(x+1)^{\frac{3}{2}}}$ c. $p(x) = 3x^2 + 4x - 5$

Solution

a. $f(x)=\frac{x^3+6x+9}{x^2+4x+1}$ is a rational function, because both $x^3 + 6x + 9$ and $x^2 + 4x + 1$ are polynomial expressions.

b. $g(x)=\frac{3x-5}{(x+1)^{\frac{3}{2}}}$ is not a rational function, because $(x + 1)^{\frac{3}{2}}$ is not a polynomial expression.

c. $p(x)=3x^2+4x-5$ is a polynomial function and any polynomial function, $p(x)$, is a rational function, because $p(x)$ can be expressed as $p(x)=\frac{p(x)}{1}$ and the denominator 1 is constant, which is a polynomial expression.

Domains of Rational Functions

Activity 2.8

For each of the following rational functions, find the number(s) that make the denominator zero.

a. $f(x)=\frac{x+1}{x-1}$

b. $g(x)=\frac{3}{x+1}$

c. $h(x)=\frac{x^2+1}{x^2-1}$

d. $l(x)=\frac{x^2-x-12}{x^2+x-6}$

For each of the rational functions in Activity 2.8, at the value of x that makes the denominator zero, the given rational function is not defined, as division by zero is not possible and such number(s) must not be in the domain of the given rational function. Thus, the definition of domain of a rational function is given as follows.

Definition 2.9

The domain of a rational function $f(x)=\frac{p(x)}{q(x)}$ is the set of all real numbers such that $q(x)$ is not zero. That is, $\text{Dom}(f)=\{x\in\mathbb{R}\mid q(x)\neq 0\}$.

Example 2

Find the domain of each of the following rational functions:

a. $f(x)=\frac{3x+5}{x^2+3x+2}$

b. $g(x)=\frac{2x^2+x-1}{x^2+x+4}$

Solution

a. Consider the denominator, x^2+3x+2 of $f(x)$..

Then, $x^2+3x+2=0 \Rightarrow x=-1$ or $x=-2$.

Therefore, $\text{Dom}(f)=\{x\in\mathbb{R}|x\neq -1 \text{ and } x\neq -2\}$.

b. Consider the denominator, x^2+x+4, of $g(x)$, which is a quadratic expression.

Since the discriminant of x^2+x+4 : $1^2-4(1)(4)=-15$ and $-15<0$, $x^2+x+4\neq 0$ for all $x\in\mathbb{R}$.

Therefore, $\text{Dom}(g)=\mathbb{R}$.

Exercise 2.9

1. Which of the following are rational functions?

   a. $f(x)=\dfrac{2x^2+3x-5}{x^3-2x^2+5x-1}$ b. $g(x)=\dfrac{2x+9}{\sqrt{x^2+x-2}}$

   c. $h(x)=\dfrac{1}{3}x^3+4x^2-x+5$

2. Find the domain of each of the following rational functions.

   a. $f(x)=\dfrac{x^2+x+5}{x^2+4x+3}$ b. $g(x)=\dfrac{x+6}{x^2+2x+2}$

2.3.2 Graphs of Rational Functions

Asymptotes to the graph of a Rational Function

Activity 2.9

Consider the function $f(x)=\dfrac{1}{x}$.

1. Complete the following tables.

x	1	0.5	0.1	0.01	0.001	0.0001	0.00001
$f(x)$							

x	−1	−0.5	−0.1	−0.01	−0.001	−0.0001	−0.00001
$f(x)$							

2. Complete the following tables.

x	1	10	100	1000	10000	100000
$f(x)$						

x	−1	−10	−100	−1000	−10000	−100000
$f(x)$						

The domain of $f(x)=\frac{1}{x}$ is $\{x \in \mathbb{R} | x \neq 0\}$ and

1. from the two tables of Activity 2.9(1), observe that;
   i. as x approaches 0 from the right, the value of the function $f(x)$ increases without bound and
   ii. as x approaches 0 from the left, the value of the function $f(x)$ decreases without bound.

Note that, x approaches to 0 from the right means, x takes numbers very close to 0, but numbers that are greater than 0 (in this case we mean positive numbers), and x approaches to 0 from the left means, x takes numbers very close to 0, but numbers less than 0 (in this case we mean negative numbers).

These two behaviors of the function $f(x)$ near $x=0$ are denoted as follows:

a. if x approaches 0 from the right (this concept is denoted by $x \longrightarrow 0^+$), then $f(x)$ increases without bound (this concept is denoted by $f(x) \longrightarrow \infty$);
b. if x approaches 0 from the left (this concept is denoted by $x \longrightarrow 0^-$), then $f(x)$ decreases without bound (this concept is denoted by $f(x) \longrightarrow -\infty$).

The line $x=0$ is called a vertical asymptote to the graph of $f(x)$.

2. From the two tables of Activity 2.9(2), observe that:
   a. as x increases without bound, $f(x)$ approaches 0 and it is expressed by:$f(x) \rightarrow 0$ as $x \rightarrow \infty$.
   b. as x decreases without bound, $f(x)$ approaches 0 and it is expressed by $f(x) \rightarrow 0$ as $x \rightarrow -\infty$.

   These two information imply that, as $x \rightarrow \infty$ or $x \rightarrow -\infty$, the graph of $f(x)$ approaches the horizontal line $y=0$.Thus, the horizontal line $y=0$ is called a horizontal asymptote to the graph of $f(x)$. With these given information, we can sketch the graph of $f(x)=\frac{1}{x}$ as in Figure 2.1

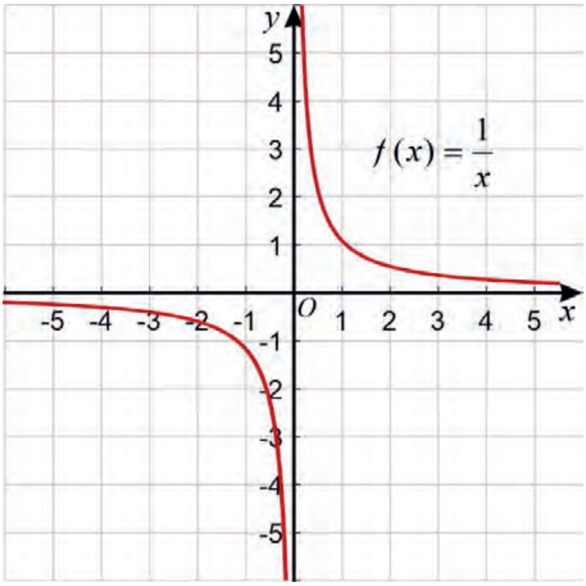


Figure 2.1: Graph of $f(x) = \frac{1}{x}$

To sketch the graph of a rational function $f(x) = \dfrac{p(x)}{q(x)}$, the following points are important:

a. the domain of the given function;

b. the behavior of the function near the points that make the denominator zero; and

c. the behavior of the function for very large positive values and for very small negative values of the variable.

Definition 2.10

Given a rational function $f(x) = \frac{p(x)}{q(x)}$;

a. If $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$ from the left or $x \to a$ from the right, then the line $x = a$ is called a vertical asymptote to the graph of $f(x)$.

b. If $f(x) \to b$ as $x \to \infty$ or as $x \to -\infty$, then the line $y = b$ is called a horizontal asymptote to the graph of $f(x)$.

Consider a rational function $f(x) = \frac{p(x)}{q(x)}$, where, $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, with $a_n \neq 0$ and $q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0$ with $b_m \neq 0$.

1. If $p(a) \neq 0$ and $q(a) = 0$, then the line $x = a$ is a vertical asymptote to the graph of $f(x)$.
2. If both $p(a) = 0$ and $q(a) = 0$, then the graph of $f(x)$ has a "hole" at $x = a$ or we have to make further simplification to decide.
3. If $n < m$, then the line $y = 0$ is a horizontal asymptote to the graph of $f(x)$.
4. If $n = m$, then the line $y = \frac{a_n}{b_n}$ is a horizontal asymptote to the graph of $f(x)$.
5. If $n = m + 1$, then the graph of $f(x)$ has an asymptote $y = ax + b, a \neq 0$, and because the line is an oblique line, $y = ax + b$ is called an oblique asymptote to the graph of $f(x)$.
6. If $n > m + 1$, the graph of $f(x)$ has no horizontal asymptote and it has no oblique asymptote.

Example 1

Find the vertical, horizontal or oblique asymptotes to the graph of each of the following functions (if any):

a. $f(x) = \frac{1}{x-2}$ b. $g(x) = \frac{x+1}{x^2+3x+2}$ c. $h(x) = \frac{3x^2+1}{x^2+4x+4}$

d. $l(x) = \frac{x^3+3x+1}{x^2+1}$ e. $m(x) = \frac{x^3+x^2+x+1}{x+2}$

Solution

a. $x - 2 = 0$ implies $x = 2$. Then, $\text{Dom}(f) = \{x \in \mathbb{R} | x \neq 2\}$.

Let $p(x) = 1$ and $q(x) = x - 2$.

Then $f(x) = \frac{1}{x-2} = \frac{p(x)}{q(x)}$.

$p(2) = 1 \neq 0$ and $q(2) = 0$.

Therefore, the line $x = 2$ is a vertical asymptote to the graph of $f(x)$.

The degree of $p(x)$ is 0 and the degree of $q(x)$ is 1. That is, the degree of $p(x)$ is less than the degree of $q(x)$. Here, the line $y = 0$ is a horizontal asymptote to the graph of $f(x)$.

b. $x^2 + 3x + 2 = 0$ implies $x = -1$ or $x = -2$.

Thus, $\text{Dom}(g) = \{x \in \mathbb{R} | x \neq -1, -2\}$.

$$g(x) = \frac{x+1}{x^2+3x+2} = \frac{x+1}{(x+2)(x+1)} = \frac{1}{x+2} \text{ for } x \neq -1.$$

Therefore, $x = -2$ is the only vertical asymptote to the graph of $g(x)$ and the line $y = 0$ is a horizontal asymptote to the graph of $g(x)$, because the degree of $x + 1$ is 1 and the degree of $x^2 + 3x + 2$ is 2. That is, the degree of $x + 1$ is less than the degree of $x^2 + 3x + 2$.

c. $x^2 + 4x + 4 = 0$ implies $x = -2$. Then the domain of $h(x)$ is $\{x \in \mathbb{R} | x \neq -2\}$.

$h(x) = \frac{3x^2+1}{x^2+4x+4} = \frac{3x^2+1}{(x+2)^2}$ and $3(-2)^2 + 1 = 7 \neq 0$.

This implies the line $x = -2$ is the vertical asymptote to the graph of $h(x)$.

Since the degree of $3x^2 + 1$ is 2 and the degree of $x^2 + 4x + 4$ is also 2, then the line $y = \frac{3}{1} = 3$ is the horizontal asymptote to the graph of $h(x)$.

d. Since $x^2 + 1 \neq 0$ for any real number x, the graph of $l(x)$ has no vertical asymptote.

Since the degree of $x^3 + 3x + 1$ is 3 and the degree of $x^2 + 1$ is 2. That is, the degree of $x^3 + 3x + 1$ is one more than the degree of $x^2 + 1$, the using long division, you can write

$$l(x) = \frac{x^3+3x+1}{x^2+1} = x + \frac{2x+1}{x^2+1}.$$

Therefore, the line $y = x$ is an oblique asymptote to the graph of $l(x)$.

e. $x+2=0$ implies $x=-2$. Thus the domain of $m(x)$ is $\{x \in \mathbb{R} | x \neq -2\}$.

$(-2)^3 + (-2)^2 + (-2) + 1 = -5 \neq 0$.

This implies $x = -2$ is a vertical asymptote to the graph of $m(x)$.

The degree of $x^3 + x^2 + x + 1$ is 3, the degree of $x + 2$ is 1 and $3 > 1 + 1$. Thus, the graph of $m(x) = \dfrac{x^3 + x^2 + x + 1}{x + 2}$ has no horizontal asymptote and it also has no oblique asymptote.

Exercise 2.10

Find the vertical, horizontal and oblique asymptote(s) (if any) to the graph of each of the following functions.

a. $f(x)=\dfrac{1}{x+2}$

b. $g(x)=\dfrac{x+2}{x^2+5x+6}$

c. $h(x)=\dfrac{x^2+3}{x^2+6x+9}$

d. $l(x)=\dfrac{x^3+2x+1}{x^2+4}$

e. $m(x)=\dfrac{x^4+2x^3-3x+1}{x-2}$

Graphs of Rational Functions

To sketch the graph of a given rational function $f(x)=\frac{p(x)}{q(x)}$ the following points are important.

1. Determine the domain of the given function.
2. Determine the point(s) at which the graph of $f(x)$ intersects x-axis, if any. These points are called x-intercepts and they can be obtained by solving the equation $p(x)=0$,
3. Determine the point at which the graph of $f(x)$ intersects y-axis, if any. This point is called y-intercept and it can be obtained by solving for y in the expression $y=f(x)$ for $x=0$ provided that, $x=0$ is in the domain of $f(x)$. If $x=0$ is not in the domain of $f(x)$, then the graph of $f(x)$ has no y-intercept.
4. Determine the vertical asymptote(s) if any. That is, find a so that $p(a)\neq 0$ and $q(a)=0$. Then the vertical line $x=a$ is a vertical asymptote to the graph of $f(x)$.
5. Determine the horizontal or oblique asymptotes (if there is any).

Example 2

Sketch the graph of the rational function $f(x) = \frac{1}{x^2}$.

Solution

Domain

$x^2 = 0 \implies x = 0$. Thus, $\text{Dom}(f) = \{x \in \mathbb{R} | x \neq 0\}$.

Intercepts

x-intercept(s): As $1 \neq 0$, the graph of $f(x)$ has no x-intercept.

y-intercept: Since $x = 0$ is not on the domain of $f(x)$, the graph of $f(x)$ has no y-intercept.

Asymptote(s)

Let $p(x) = 1$ and $q(x) = x^2$. Then, $f(x) = \frac{p(x)}{q(x)}$:

i. $p(0) = 1$ and $q(0) = 0$ and this implies $x = 0$ is a vertical asymptote to the graph of $f(x)$.

ii. The degree of $p(x)$ is 0 and the degree of $q(x)$ is 2, that is, the degree of $p(x)$ is less than the degree of $q(x)$, the line $y = 0$ is a horizontal asymptote to the graph of $f(x)$.

You can use the following table to sketch the graph of $f(x)$.

x	-10	-1	-0.1	0.1	1	10
$f(x)$	0.01	1	100	100	1	0.01

The graph of $f(x)$ is given in **Figure 2.2**.

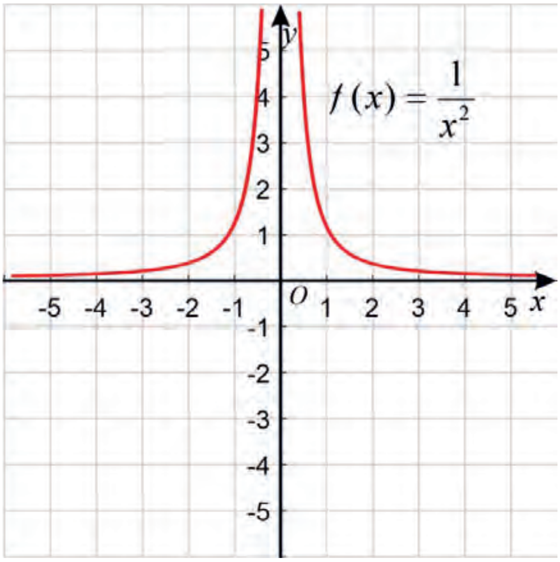


Figure 2.2: Graph of $f(x) = \frac{1}{x^2}$

Example 3

Sketch the graph of the rational function $g(x) = \frac{x+1}{x^2+5x+6}$.

Solution

Domain:

$x^2 + 5x + 6 = 0$ implies $x = -2$ or $x = -3$.

Thus, $\text{Dom}(g) = \{x \in \mathbb{R} | x \neq -2, -3\}$.

Intercepts:

x-intercept(s): $x+1=0$ implies $x = -1$.

This implies, $(-1,0)$ is the x-intercept of the graph of $g(x)$.

y-intercept: $g(0) = \frac{0+1}{0^2+5(0)+6} = \frac{1}{6}$.

This implies, $\left(0, \frac{1}{6}\right)$ is the y-intercept of the graph of $g(x)$.

Asymptote(s):

Vertical asymptote(s):

Observe that $-2+1=-1\neq 0$, $(-2)^2+5(-2)+6=0$, $-3+1=-2\neq 0$ and $(-3)^2+5(-3)+6=0$.

This implies, both $x=-2$ and $x=-3$ are vertical asymptotes to the graph of $g(x)$.

Horizontal/Oblique Asymptote:

Since the degree of $x+1$ is 1, the degree of the degree of x^2+5x+6 is 2 and $1<2$, the line $y=0$ is a horizontal asymptote to the graph of $g(x)$.

You can use the following table to sketch the graph of $g(x)$.

x	-4	0	1	2
$g(x)$	$-\frac{3}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{3}{20}$

The graph of $g(x)$ is given in **Figure 2.3**.

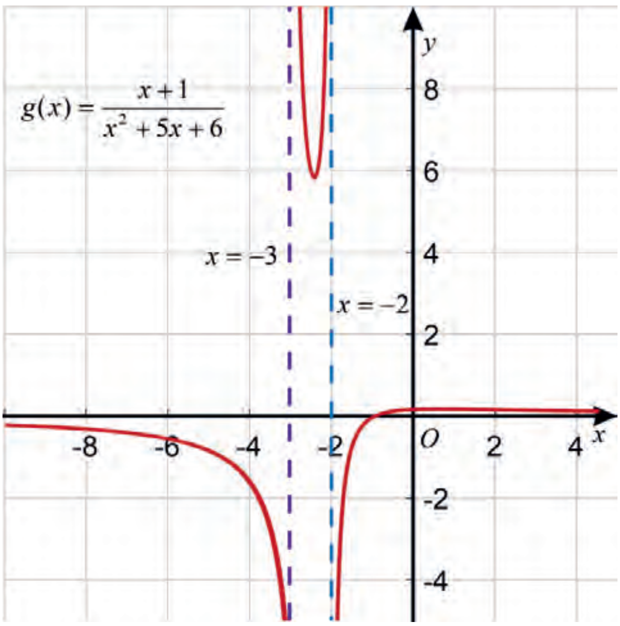


Figure 2.3: Graph of $g(x)=\dfrac{x+1}{x^2+5x+6}$

Example 4

Sketch the graph of the rational function $h(x)=\dfrac{x^2}{x^2+3x+2}$.

Solution

Domain:

$x^2+3x+2=0$ implies $x=-1$ or $x=-2$.

Then, $\text{Dom}(h)=\{x\in\mathbb{R}|x\neq -1,-2\}$.

Intercepts:

***x*-intercept(s):** $x^2=0$ implies $x=0$.

This implies, $(0,0)$ is the x-intercept of the graph of $h(x)$.

***y*-intercept:** $\dfrac{0^2}{0^2+3(0)+2}=0$ implies $(0,0)$ is the y-intercept of the graph of $h(x)$.

Asymptotes:

Vertical Asymptote(s):

- $(-1)^2=1\neq 0$ and $(-1)^2+3(-1)+2=0$.

This implies the line $x=-1$ is a vertical asymptote to the graph of $h(x)$.

- $(-2)^2=4\neq 0$ and $(-2)^2+3(-2)+2=0$.

This implies the line $x=-2$ is vertical asymptote to the graph of $h(x)$.

Horizontal/Oblique Asymptote:

Since the degree of x^2 is 2, the degree of x^2+3x+2 is 2 and $2=2$, the line $y=\frac{1}{1}=1$ is the horizontal asymptote to the graph of $h(x)$.

We can use the following table to sketch the graph of $h(x)$.

x	0	1	2	3	4	5
$h(x)$	0	$0.1\dot{6}$	$0.\dot{3}$	0.45	$0.5\dot{3}$	≈ 0.595

Then, the graph of $h(x)$ is given in **Figure 2.4.**

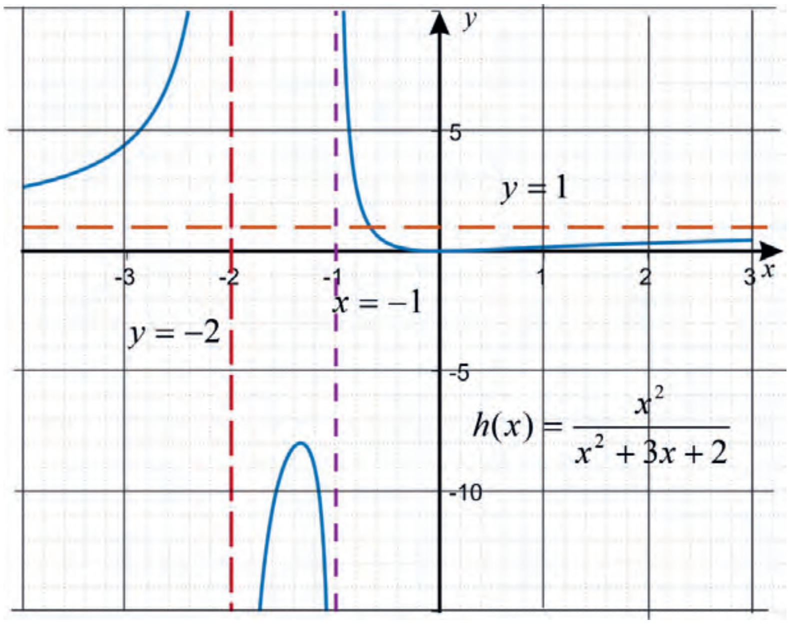


Figure 2.4: Graph of $h(x) = \frac{x^2}{x^2+3x+2}$

Example 5

Sketch the graph of the rational function $l(x) = \frac{x^3+1}{x^2+4x+4}$.

Solution

Domain:

$x^2 + 4x + 4 = 0$ implies $(x+2)^2 = 0$.

This implies, $x = -2$

Thus, $\text{Dom}(l) = \{x \in \mathbb{R} \mid x \neq -2\}$.

Intercepts:

***x*-intercept(s):** $x^3 + 1 = 0$ implies $x = -1$.

This implies $(-1, 0)$ is the x-intercept of the graph of $l(x)$.

***y*-intercept:** For $x = 0$, $h(0) = \frac{1}{4}$ and hence $\left(0, \frac{1}{4}\right)$ is the y-intercept of the graph of $l(x)$.

Asymptotes

Vertical Asymptote(s)

$(-2)^3 + 1 = -8 + 1 = -7 \neq 0$ and $(-2)^2 + 4(-2) + 4 = 0.$

This implies, the line $x = -2$ is a vertical asymptote to the graph of $l(x)$.

Horizontal/Oblique Asymptote:

The degree of $x^3 + 1$ is 3, the degree of $x^2 + 4x + 4$ is 2 and $3 > 2$.

Then, using long division we can write:

$$\frac{x^3+1}{x^2+4x+4} = (x-4) + \frac{12x+17}{x^2+4x+4}$$

Therefore, the line $y = x - 4$ is an oblique asymptote to the graph of $l(x)$.

We can use the following table to sketch the graph of $l(x)$.

x	-3	-1	0	2
$g(x)$	-24	0	$\frac{1}{4}$	$\frac{9}{16}$

The graph of l is given in **Figure 2.5.**

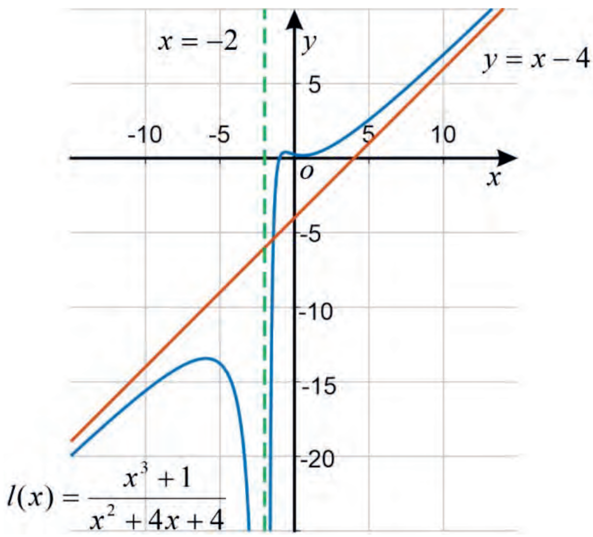


Figure 2.5: Graph of $l(x) = \frac{x^3+1}{x^2+4x+4}$

Exercise 2.11

Sketch the graphs of each of the following rational functions.

a. $f(x)=\dfrac{4}{x+2}$

b. $g(x)=\dfrac{5x}{6x-2}$

c. $h(x)=\dfrac{1}{x^2+x-12}$

d. $l(x)=\dfrac{2x-1}{-2x^2-5x+3}$

e. $m(x)=\dfrac{x^2+1}{x+3}$

2.4 Applications

Rational formulas can be useful tools for representing real-life situations and for finding answers to real problems. Equations representing direct, inverse, and joint variation are examples of rational formulas that can model many real-life situations.

The following steps are used in solving word problems.

Step 1: Understanding the problem

Step 2: Setting up the equation(s)/or expressions using variables.

Step 3: Solving the given equation(s) for the given variables.

Step 4: Interpreting the results.

Example 1

Person A takes 2 hours to plant 50 flower bulbs and person B takes 3 hours to plant 45 flower bulbs. If they are working together, how long should it take them to plant 160 bulbs?

Solution

To solve this problem, consider the planting rate of the two persons; that is, how many bulbs each person can plant in one hour.

A: $\dfrac{50 \text{ Bulbs}}{2 \text{ Hours}}=\dfrac{25 \text{ Bulbs}}{1 \text{ Hour}}$ and B: $\dfrac{45 \text{ Bulbs}}{3 \text{ Hours}}=\dfrac{15 \text{ Bulbs}}{1 \text{ Hour}}$

Now, combine the hourly rates of the two persons to determine the rate of the work when they work together.

A and B together: $\frac{25 \text{ Bulbs}}{1\text{Hour}} + \frac{15 \text{ Bulbs}}{1 \text{ Hour}} = \frac{40 \text{ Bulbs}}{1 \text{ Hour}}$.

That is, when A and B work together, they plant 40 flower bulbs in an hour.

From the work formula, $W = rt$, we have $r = \frac{W}{t}$, where r is the combined work rate, W is the amount of work that must be done and t is the amount of time it will take to complete the work.

The required information is, how much time it will take to do the required work at the given rate. To do so we have to solve the rational equation.

$$\frac{40}{1} = \frac{160}{t}.$$

Solving this equation for t gives us that t = 4 hours.

Thus, if A and B work together, it will take 4 hours for them to plant 160 bulbs together.

Example 2

A sample of 80 g of a particular ice cream contains 15 gram of fat and 20 g of carbohydrate.

a. How much fat does 350 g of the same ice cream contain?

b. How much carbohydrate does 350 g of the same ice cream contain?

Solution

a. If the amount of fat in 350 g is x g, then $\frac{x}{350} = \frac{15}{80}$..

Solving for x gives us, $x = \frac{15}{80} \times 350$ g= 65.625 g

b. If the amount of carbohydrate in 350 g is y g, then $\frac{y}{350} = \frac{20}{80}$.

Solving for y gives us, $y = \frac{20}{80} \times 350\text{g} = 87.5\text{g}$.

Example 3

A large mixing tank contains 100 gallons of water into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank. At the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after 10 minutes. Compare the concentration after 10 minutes with the concentration at the beginning.

Solution

Let t be the number of minutes since the tap opened. Since the water increases at 10 gallons per minute, and the sugar increases at 1 pound per minute, these are constant rates of change, that is,

$W(t) = 100 + 10t$ in gallons (the amount of water in the tank after t minutes) and

$S(t) = 5 + t$ in pounds (the amount of sugar in the tank after t minutes).

The concentration C will be the ratio of pounds of sugar to gallons of water.

That is, $C(t) = \frac{5+t}{100+10t}$.

Thus, the concentration after 10 minutes is given by $C(10) = \frac{5+10}{100+10(10)} = \frac{15}{200} = 0.075$.

The concentration at the beginning was $C(0) = \frac{5}{100} = 0.05$.

Since $0.075 > 0.05$, there is high concentrations after 10 minutes than at the beginning.

Exercise 2.12

1. Two pipes can fill a tank in 6 hr. The larger pipe works twice as fast as the smaller pipe. How long would it take each pipe to fill the tank if they worked separately?
2. Legesse bought USD 150 with Birr 7500. At this rate, how many USD can he buy with Birr 30000?
3. Sofia drove her car 45 kilometers and she ran out of gas and had to walk 2 kilometers to a gas station. Her speed of driving is 15 times her speed of walking. If the total time for the drive and walk was $1\frac{1}{2}$ hours, what was Sofia's driving speed?

Problem Solving

1. Zeru can complete a certain job in 5 days and Tulu can complete the same job in 3 days. How long will it take them to complete the job if they work together?
2. In one morning, if the shadow of a building of 20 meters long is 15 meters long, how long is the shadow of a tree of 10 meters long?

Summary

1. Let $p(x)$ and $q(x)$ be two polynomial expressions. An expression of the form $\frac{p(x)}{q(x)}$, where $q(x) \neq 0$, is a rational expression

2. Given two rational expressions $\frac{p(x)}{q(x)}$ and $\frac{r(x)}{s(x)}$ with $q(x) \neq 0$ and $s(x) \neq 0$, we define

   a. $\frac{p(x)}{q(x)} + \frac{r(x)}{s(x)} = \frac{p(x)s(x)+r(x)q(x)}{q(x)s(x)}$

   b. $\frac{p(x)}{q(x)} - \frac{r(x)}{s(x)} = \frac{p(x)s(x)-r(x)q(x)}{q(x)s(x)}$

   c. $\frac{p(x)}{q(x)} \times \frac{r(x)}{s(x)} = \frac{p(x)r(x)}{q(x)s(x)}$

   d. $\frac{p(x)}{q(x)} \div \frac{r(x)}{s(x)} = \frac{p(x)}{q(x)} \times \frac{s(x)}{r(x)} = \frac{p(x)s(x)}{q(x)r(x)}, (r(x) \neq 0)$

3. A function f that can be expressed in the form $f(x) = \frac{p(x)}{q(x)}$, where both $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$, is called a rational function.

4. The domain of a rational function $f(x) = \frac{p(x)}{q(x)}$ is the set of all real numbers such that $q(x)$ is not zero.

5. Given a rational function $f(x) = \frac{p(x)}{q(x)}$;

   a. If $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$ from the left or $x \to a$ from the right, then the line $x = a$ is called a vertical asymptote to the graph of $f(x)$.

   b. If $f(x) \to b$ as $x \to \infty$ or as $x \to -\infty$, then the line $y = b$ is called a horizontal asymptote to the graph of $f(x)$.

Review Exercise

1. Find the domain and simplify each of the following rational expressions.

a. $\frac{x-1}{x^2-1}$ b. $\frac{x^2-5x+6}{x^2-4}$

c. $\frac{4x^2-36}{x^2-6x+9}$ d. $\frac{x^2+1}{x^3+2x^2+x}$

2. Find the domain and perform each of the following indicated operations.

a. $\frac{x-1}{x+2}+\frac{x+3}{x^2+6x+5}$ b. $\frac{x-1}{x+2}-\frac{x+3}{x^2+6x+5}$

c. $\frac{(x-1)}{(x+2)}\times\frac{(x+3)}{(x^2+6x+5)}$ d. $\frac{x-1}{x+2}\div\frac{x+3}{x^2+6x+5}$

3. Write each of the following rational expressions as sum of partial fractions.

a. $\frac{5x+6}{x^2-4}$ b. $\frac{x+6}{x^2-4x+4}$

c. $\frac{6x+5}{x^4+4x^2}$ d. $\frac{x^2+x+2}{x^2+6x+8}$ e. $\frac{x^3+2x^2+2}{x^2-x-6}$

4. Solve each of the following rational equations.

a. $x+\frac{4}{x+3}=2$ b. $1-\frac{5}{x}-\frac{6}{x^2}=0$ c. $\frac{3x}{x+4}+\frac{2}{x^2+6x+8}=\frac{1}{x+2}$

5. Solve each of the following rational inequalities.

a. $\frac{x+3}{x-1}>0$ b. $\frac{2x}{x-4}\leq 1$ c. $\frac{1}{3}-\frac{2}{x^2}\geq\frac{5}{3x}$ d. $\frac{5}{x-1}\geq\frac{4}{x+2}$

6. For each of the following rational functions:

   i. determine the domain;

   ii. find the x-intercept(s) and y-intercept of the graph of the function, if any;

   iii. find the vertical asymptote(s) and horizontal/oblique asymptote to the graph of the function, if any;

iv. sketch the graph of each of the following functions.

a. $f(x)=\dfrac{1}{x-4}$ b. $g(x)=\dfrac{x+2}{x^2+5x+6}$

c. $h(x)=\dfrac{x^2+1}{x^2-4x+4}$ d. $k(x)=\dfrac{x+6}{x^2+4}$ e. $l(x)=\dfrac{x^3+6x^2+5x}{x^2-4x+4}$

7. Let The ratio of boys to girls in a certain high school is 4: 5. If the total number of students in the school is 1260, find the number of boys and the number of girls in the school.

UNIT 3

MATRICES

Unit Outcomes

By the end of this unit, you will be able to:

- Know basic concepts about matrices.
- Perform operation on matrices.
- Differentiate types of a matrix.
- Know specific ideas, methods and principles concerning matrices.
- Formulate elementary row/column operations.
- Define inverse of an invertible matrix.
- Use elementary row operations to find inverse of square matrices of order 2 and 3.
- Define system of linear equations.
- Solve linear system of equations by using elementary row operations.
- Apply matrix concepts to model, solve and analyze real-world situations.

Unit Contents

KEY Words

- augmented matrix
- column matrix
- consistent system
- diagonal matrix
- elementary row operations
- inconsistent system
- inverse matrix
- non-singular matrix
- reduced-echelon form
- row echelon form
- row matrix
- scalar matrix
- singular matrix
- skew-symmetric matrix
- symmetric matrix
- transpose of a matrix
- triangular matrix
- zero matrix

Introduction

Systems of linear equations occur in many areas such as in Geometry, Engineering, Business, Economics, Biology, etc. Solving such systems of linear equations is one of the most important applications of mathematics to other fields.

In Unit 3 of Grade 9, you have learned how to solve systems of linear equations in two variables. In this unit, you will learn how to solve systems of linear equations in a more systematic way.

Consider the following two linear equations in the variables x and y, $x - y = 1$ and $2x + y = 7$. A common solution for these two equations can be found by working only on the numbers involved in the two equations. These numbers form a rectangular array of numbers as follows:

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 7 \end{pmatrix}$$

Such array of numbers is called a matrix. Matrices (plural form of a matrix) are involved in different applications of mathematics.

3.1 The Concepts of a Matrix

Activity 3.1

1. Consider the seats in your classroom.
   a. How many rows of seats are there in your classroom?
   b. How many columns of seats are there is your classroom?
   c. For each of the students in your class, identify the row and column that the student is found.
2. Consider the following table, which gives the sections and the numbers of male and female students in each section of Grade 11 students of a certain school.

	Male	Female
Section 1	25	26
Section 2	28	24
Section 3	30	20
Section 4	15	35

   a. Find the number of female students in Sections 1, 2, 3 and 4.
   b. Find the number of male students in Sections 1, 2, 3 and 4.

In most classrooms, seats are arranged in rows and columns. In such arrangements, one can assign an ordered pair of numbers to each seat which indicates the position of the row and the position of the column that the seat is found.

The information in Activity 3.1 (2), can be displayed as an array of numbers as follows:

$$\begin{pmatrix} 25 & 26 \\ 28 & 24 \\ 30 & 20 \\ 15 & 35 \end{pmatrix}.$$

This array of numbers is a matrix and a formal definition of matrices is given below.

Definition 3.1

Let m and n be positive integers. A rectangular array of numbers A in $\mathbb{R}$ that is written as:

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{pmatrix}$$

is called an m by n matrix (or $m \times n$) matrix in $\mathbb{R}$.

In Definition 3.1:

a. the number m is the number of rows, n is the number of columns and $m \times n$ is called the size of the matrix and it is read as "m by n".

b. The number a_{ij} is called the ij-entry or ij -element of A. That is, numbers in the given array are called elements or entries of A. The ij -entry is an element appearing in the i^{th} row and the j^{th}-column of the matrix.

c. Matrix A can be abbreviated by writing as $A = (a_{ij})_{m\times n}$.

d. Matrices are usually denoted by capital letters like A, B, C etc and entries of matrices by small letters like a, b, c etc.

Example 1

Find the size and determine its elements/or entries for each of the following matrices.

a. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

b. $B = \begin{pmatrix} -2 & 10 \\ 3 & -1 \\ 0 & 5 \end{pmatrix}$

Solution

a. A has two rows and two columns. Therefore, A is a 2×2 matrix.

The entries of A are:

i. $a_{11} = 1$ is the entry found in the first row and in the first column of A;

ii. $a_{12} = 2$ is the entry found in the first row and in the second column of A;

iii. $a_{21} = 3$ is the entry found in the second row and in the first column of A; and

iv. $a_{22} = 4$ is the entry found in the second row and in the second column of A.

b. B has three rows and two columns. Therefore, B is a 3×2 matrix.

The entries of B are:

i. $b_{11} = -2$ is the entry found in the first row and in the first column of B;

ii. $b_{12} = 10$ is the entry found in the first row and in the second column of B;

iii. $b_{21} = 3$ is the entry found in the second row and in the first column of B;

iv. $b_{22} = -1$ is the entry found in the second row and in the second column of B;

v. $b_{31} = 0$ is the entry found in the third row and in the first column of B; and

vi. $b_{32} = 5$ is the entry found in the third row and in the second column of B.

Example 2

Find the size of the matrix and determine its elements/or entries for each of the following matrices:

a. $C = \begin{pmatrix} 1 \\ 0 \\ -2 \\ \frac{1}{4} \end{pmatrix}$. b. $D = \left(2, 1, 3, 0, -1, \frac{1}{4}\right)$ c. $E = (-4)$.

Solution

a. Matrix C has four rows and one column. Therefore, C is a 4×1 matrix.

The entries of C are:

i. $c_{11} = 1$ is the entry found in the first row and in the first column of C;

ii. $c_{21} = 0$ is the entry found in the second row and in the first column of C;

iii. $c_{31} = -2$ is the entry found in the third row and in the first column of C; and

iv. $c_{41} = \frac{1}{4}$ is the entry found in the fourth row and in the first column of C.

b. Matrix D has one row and six columns. Therefore, D is a 1×6 matrix.

The entries of D are:

i. $d_{11} = 2$ is the entry found in the first row and in the first column of D;

ii. $d_{12} = 1$ is the entry found in the first row and in the second column of D;

iii. $d_{13} = 3$ is the entry found in the first row and in the third column of D;

iv. $d_{14} = 0$ is the entry found in the first row and in the fourth column of D;

v. $d_{15} = -1$ is the entry found in the first row and in the fifth column of D; and

vi. $d_{16} = \frac{1}{4}$ is the entry found in the first row and in the sixth column of D.

c. Matrix E has one row and one column. Therefore, E is a 1×1 matrix.

The entry of E is $e_{11} = -4$, the entry found in the first row and in the first column of E.

Exercise 3.1

1. Let $A=\begin{pmatrix}1 & 2 & 1 & 0\\ 3 & 4 & 5 & -1\\ 2 & 2 & 7 & 10\end{pmatrix}$. Then determine each of the following entries (if it exists).

   a. a_{11} b. a_{23} c. a_{32} d. a_{41} e. a_{14}

2. Determine the size and the elements/or entries of each of the following matrices

   a. $A=\begin{pmatrix}2 & 3\\ 3 & 1\\ 1 & 1\end{pmatrix}$ b. $B=\begin{pmatrix}1 & 2 & 1\\ -1 & 4 & 0\end{pmatrix}$ c. $C=\begin{pmatrix}1 & 2 & 1 & 3\end{pmatrix}$

   d. $D=\begin{pmatrix}1\\ 1\\ 2\\ 2\end{pmatrix}$ e. $E=(5)$

3. Let $A=(a_{ij})_{2\times3}$ and $a_{ij}=j-i$. Then determine A.

3.2 Operations on Matrices

Applications of matrices involve different operations on matrices and some of these operations are addition, subtraction and multiplication of matrices by scalars and matrix multiplications. These operations will be considered in this section.

3.2.1 Equality of Matrices

Activity 3.2

Consider the matrices $A=\begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix}$, $B=\begin{pmatrix}3 & 2\\ 1 & 4\end{pmatrix}$ and $C=(c_{ij})_{2\times2}$, where $c_{ij}=i$ for $i=1,2$ and $j=1,2$.

a. Compare each of the corresponding entries of A and B.

b. Compare each of the corresponding entries of A and C.

From your responses in Activity 3.2, you have observed that matrices A and C have the same size and the corresponding entries of A and C are equal and such matrices are called equal matrices.

For most operations on matrices, equality of two matrices is required and it is defined as follows.

Definition 3.2

Two matrices $A=(a_{ij})_{m\times n}$ and $B=(b_{ij})_{m\times n}$ with the same size are said to be equal, written as A=B, if their corresponding entries are equal. That is, A = B, if $a_{ij} = b_{ij}$ for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

Let $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. If A = B, then $a_{11}=b_{11}$, $a_{12}=b_{12}$, $a_{21}=b_{21}$ and $a_{22}=b_{22}$.

Example

Find the values of the unknowns in each of the following, if A = B.

1. $A=\begin{pmatrix} 1 & 2 \\ -3 & a \end{pmatrix}$ and $B=\begin{pmatrix} 1 & b \\ -3 & 4 \end{pmatrix}$
2. $A=\begin{pmatrix} a+b & -2 \\ 5 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} 3 & -2 \\ 5 & a-b \end{pmatrix}$
3. $A=\begin{pmatrix} a+b & 5 & e \\ 3 & -1 & 4 \\ 0 & 1 & 9 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 5 & 7 \\ a-b & -1 & 4 \\ d & 1 & c \end{pmatrix}$

Solution

1. If A = B, then the corresponding entries of the two matrices are equal.

   That is, $a = 4$ and $b = 2$.
2. If A = B, then the corresponding entries of the two matrices are equal.

That is, $a+b=3$ and $a-b=1$. Adding the two equations gives you $2a = 4 \Longrightarrow a = 2$.

Then solving for b gives you $b = 3-2 = 1$.

3. If A = B, then the corresponding entries of the two matrices are equal.

This implies $a+b=1$, $a-b=3$, $c = 9$, $d = 0$ and $e = 7$.

Then let us solve the two equations

$$\begin{array}{l} a+b=1 \\ a-b=3 \end{array}.$$

Adding the two equations gives $2a = 4$ which implies $a = 2$. Then substituting $a = 2$ in either of the equations $a+b=1$ or $a-b=3$ and solving for b gives us $b = -1$.

Therefore, if A = B, then $a = 2, b = -1, c = 9$, $d = 0$ and $e = 7$.

Note that equality of matrices are defined only when the two matrices have the same size.

Exercise 3.2

1. Let $A = \begin{pmatrix} 3 & a & 2 \\ b & 1 & 5 \\ 2 & 4 & c \end{pmatrix}$ and $B = \begin{pmatrix} d & 5 & 2 \\ 4 & e & 5 \\ f & 4 & 6 \end{pmatrix}$. If A = B, then determine the values of a, b, c, d, e and f.

2. Let $A = \begin{pmatrix} 8 & a+b \\ 5 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 8 & 3 \\ 5 & a-b \end{pmatrix}$. If $A = B$, then determine the values of a and b.

3. Let $A=\begin{pmatrix} a+b & 2 & 3 \\ a-b & 0 & -1 \\ 4 & 5 & 6 \end{pmatrix}$ and $B=\begin{pmatrix} 2 & 2 & c \\ 4 & 0 & -1 \\ 4 & d & 6 \end{pmatrix}$. If $A=B$, then determine the values of a, b, c and d.

3.2.2 Additions of Matrices

Activity 3.3

The following two tables give the numbers of child and adult shoes of both sexes, male and female, of a certain company in two different shops: Shop A and Shop B.

Number of Shoes in Shop A

	Male	Female
Child	65	42
Adult	111	154

Number of Shoes in Shop B

	Male	Female
Child	15	21
Adult	19	28

a. Find the number of child and adult shoes of both sexes that the company has in both shops and present your solutions in a table form.

b. The following table gives the number of shoes of each type the company sells in a particular week from Shop A. Find the number of shoes of each type remaining in Shop A at the end of the week.

Shop A

	Male	Female
Child	33	20
Adult	90	120

In Activity 3.3, observe that,

a. to obtain the number of shoes of each type in both shops, you have added the corresponding entries in the two tables;

b. to obtain the number of shoes of each type in Shop A after a sell in the given week, you have subtracted the corresponding entries in the two tables in the proper order.

This leads us to define the first operation on matrices, addition of matrices.

Addition of two matrices is defined provided that the two matrices have the same size. Thus, the sum of the two matrices is a matrix of the same size as the two given matrices and its entries are obtained by adding the corresponding entries of the two matrices.

Definition 3.3

Let m and n be positive integers and $A=(a_{ij})_{m\times n}$ and $B=(b_{ij})_{m\times n}$ be matrices in $\mathbb{R}$. Then

a. the sum of A and B, denoted by A + B, is an $m\times n$ matrix obtained by adding the corresponding entries of A and B.

That is, $A+B=(a_{ij}+b_{ij})_{m\times n}$

b. the subtraction of B from A, denoted by $A-B$, is an $m\times n$ matrix obtained by subtracting the entries of B from the corresponding entries of A.

That is, $A-B=(a_{ij}-b_{ij})_{m\times n}$.

Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B=\begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Then,

i. $A+B=\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$ ii. $A-B=\begin{pmatrix} a-e & b-f \\ c-g & d-h \end{pmatrix}$

Example 1

Let $A=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B=\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$. Then, compute A+B and A − B.

Solution

Both A and B are 2×2 matrices and hence both $A + B$ and $A - B$ are defined.

Thus,

a. $A + B = \begin{pmatrix} 1+0 & 2-1 \\ 3+1 & 4+2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 6 \end{pmatrix}$.

b. $A - B = \begin{pmatrix} 1-0 & 2-(-1) \\ 3-1 & 4-2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$

Example 2

Let $A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & 3 \\ 4 & 1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 & 5 \\ 1 & -4 & 3 \\ 0 & 0 & -7 \end{pmatrix}$. Then compute A+B and A − B.

Solution

Both A and B are 3×3 matrices and hence both $A + B$ and $A - B$ are defined.

Thus,

a. $A+B = \begin{pmatrix} 1-1 & -1+2 & 2+5 \\ 0+1 & 2-4 & 3+3 \\ 4+0 & 1+0 & 5-7 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 7 \\ 1 & -2 & 6 \\ 4 & 1 & -2 \end{pmatrix}$

b. $A-B = \begin{pmatrix} 1-(-1) & -1-2 & 2-5 \\ 0-1 & 2-(-4) & 3-3 \\ 4-0 & 1-0 & 5-(-7) \end{pmatrix} = \begin{pmatrix} 2 & -3 & -3 \\ -1 & 6 & 0 \\ 4 & 1 & 12 \end{pmatrix}$

Activity 3.4

Let $A = \begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 3 & -2 \\ 4 & 5 & 2 \end{pmatrix}$, $0_{2\times2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and

$0_{2\times3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then find

a. $A + 0_{2\times2}$

b. $0_{2\times2} + A$

c. $B + 0_{2\times3}$

d. $0_{2\times3} + B$

From Activity 3.4, observe that,

(i) $A + 0_{2\times2} = A = 0_{2\times2} + A$;

(ii) $B + 0_{2\times3} = B = 0_{2\times3} + B$.

Thus, matrices $0_{2\times2}$ and $0_{2\times3}$ are called the zero matrices of their respective order and do not change matrices on matrix addition.

Definition 3.4

Let m and n be positive integers. Consider the $m \times n$ matrix whose all entries are zero and we denote this matrix by $0_{m\times n}$ or simply by 0 if the order is clear.

That is, $0_{m\times n} = \begin{pmatrix} 0 & 0 \ldots & 0 \\ 0 & 0 \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 \ldots & 0 \end{pmatrix}_{m\times n}$ and it is called the zero matrix.

Observe that, for any an $m \times n$ matrix $A = (a_{ij})_{m\times n}$, if 0 denotes the $m \times n$ zero matrix, then $A + 0 = (a_{ij} + 0)_{m\times n} = (a_{ij})_{m\times n} = (0 + a_{ij})_{m\times n} = 0 + A = A$.

Therefore, the zero matrix is the identity matrix for matrix addition.

Exercise 3.3

1. Let $A = \begin{pmatrix} 1 & 3 \\ -2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. Compute each of the following matrices.
   a. $A + B$ b. $A + C$ c. $A - B$ d. $A - C$
2. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 & 4 \\ 4 & 0 & -1 \\ 5 & 2 & -2 \end{pmatrix}$. Compute each of the following matrices.
   a. $A + B$ b. $A - B$ c. $B - A$ d. $A + 0$ e. $0 + B$

3.2.3 Properties of Matrix Addition

Activity 3.5

Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 1 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & -2 \\ 1 & -5 \end{pmatrix}$. Then find

a. A + B   b. B+A   c. A+(B+C)   d. (A+B)+C

From Activity 3.5, you observed that:

(i) A+B = B+A; that is, addition of the two given matrices is commutative.

(ii) A+(B+C)=(A+B)+C; that is, addition of the given three matrices is associative.

In general, addition of matrices is both commutative and associative.

Commutative and Associative Properties of Matrix Addition

Let m and n be positive integers and A, B and C be $m \times n$ matrices in $\mathbb{R}$. Then

a. A + B = B + A (Addition of Matrices is Commutative);

b. A + (B + C) = (A + B) + C (Addition of Matrices is Associative).

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -2 \\ 2 & -5 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & -1 \\ -2 & 5 \end{pmatrix}$. Then determine

a. A + B   b. B+A   c. A+(B+C)   d. (A+B)+C

Solution

a. $A + B = \begin{pmatrix} 1+1 & 1+(-2) \\ 3+2 & 1+(-5) \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}$

b. $B + A = \begin{pmatrix} 1+1 & -2+1 \\ 2+3 & -5+1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}$.

c. $B+C=\begin{pmatrix}1+1 & -2+(-1)\\ 2+(-2) & -5+5\end{pmatrix}=\begin{pmatrix}2 & -3\\ 0 & 0\end{pmatrix}$ and

$A+(B+C)=\begin{pmatrix}1+2 & 1+(-3)\\ 3+0 & 1+0\end{pmatrix}=\begin{pmatrix}3 & -2\\ 3 & 1\end{pmatrix}$.

d. $A+B=\begin{pmatrix}2 & -1\\ 5 & -4\end{pmatrix}$ and $(A+B)+C=\begin{pmatrix}2+1 & -1+(-1)\\ 5+(-2) & -4+5\end{pmatrix}=\begin{pmatrix}3 & -2\\ 3 & 1\end{pmatrix}$

From (a) and (b), observe that $A+B=B+A$ and from (c) and (d) observe also that $A+(B+C)=(A+B)+C$.

Exercise 3.4

Let $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$, $B=\begin{pmatrix}2 & 3\\ -1 & -2\end{pmatrix}$ and $C=\begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix}$. Determine each of the following matrices.

a. $A+B$ b. $B+A$ c. $(A+B)+C$ d. $A+(B+C)$

3.2.4 Scalar Multiplication of Matrices

Activity 3.6

The following table gives the marks of Nigisti and Tadesse out of 50 in three subjects: English, Mathematics and Biology.

	Nigisti	Tadesse
English	36	34
Mathematics	40	42
Biology	38	39

a. Convert the mark of each subject of the two students out of 100.

b. Convert the mark of each subject of the two students in the three subjects out of 25.

In Activity 3.6, to obtain the marks of both students in the three subjects, you have to multiply

a. each mark of both students by 2 for the first question;

b. each mark of both students by $\frac{1}{2}$ for the second question.

In the following definition, multiplying a matrix by a number or scalar is defined.

Definition 3.5

Let $A = (a_{ij})_{m \times n}$ be a matrix in $\mathbb{R}$ and $k \in \mathbb{R}$. The scalar multiple of A by k, denoted by kA, is the $m \times n$ matrix defined by:

$$kA = k\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}.$$

That is, the entries of kA are obtained by multiplying each entry of A by k.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and k is a scalar, then $kA = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$.

Example

Let $A = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 1 \\ -2 & -1 & 3 \end{pmatrix}$. Then find 3A and $-\frac{2}{3}$B.

Solution

a. $3A = \begin{pmatrix} 3\times 1 & 3\times 0 \\ 3\times 2 & 3\times(-3) \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & -9 \end{pmatrix}$.

b. $-\frac{2}{3}B = \begin{pmatrix} \left(-\frac{2}{3}\right)\times 2 & \left(-\frac{2}{3}\right)\times 1 & \left(-\frac{2}{3}\right)\times 1 \\ \left(-\frac{2}{3}\right)\times(-2) & \left(-\frac{2}{3}\right)\times(-1) & \left(-\frac{2}{3}\right)\times 3 \end{pmatrix} = \begin{pmatrix} -\frac{4}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \frac{4}{3} & \frac{2}{3} & -2 \end{pmatrix}$

Activity 3.7

Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$. Then find,

a. $3(A + B)$
b. $3A+3B$
c. $2(3A)$
d. $(2 \times 3)A$
e. $(2 + 3)A$
f. $2A + 3A$
g. $1A$

From your solutions in Activity 3.7, observe that,

a. $3(A + B) = 3A+3B$
b. $2(3A) = (2 \times 3)A$
c. $(2 + 3)A = 2A + 3A$
d. $1A = A$

These properties are true for any two matrices A and B of the same size and for any two scalars r and s.

Properties of Scalar Multiplication of Matrices

Let m and n be positive integers, A and B be $m \times n$ matrices and $r, s \in \mathbb{R}$. Then:

a. $r(A + B) = rA + rB$. (Scalar multiplication is distributive over additions of matrices.)
b. $(r + s)A = rA + sA$. (Scalar multiplications are distributive over additions of numbers.)
c. $(rs)A = r(sA)$
d. $1A = A$

Scalar Multiplication of Matrices and Subtraction of Matrices

If A and B are $m \times n$ matrices, then

i. $(-1)A = -A$ and $A + (-A) = 0_{m\times n} = (-A) + A$. This implies $-A$ is additive inverse of A;
ii. the subtraction of two matrices A and B, $A - B$, is given by $A - B = A + (-1)B$.

Exercise 3.5

1. Let $A=\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}$ and $B=\begin{pmatrix}3 & -1\\ 2 & -4\end{pmatrix}$. Then determine each of the following.

   a. $4A$ b. $-\frac{1}{3}B$ c. $2A+3B$

   d. matrix C such that $A+2C=B$ e. $A-\frac{1}{2}B$ f. $3A-2B$

2. Let $A=\begin{pmatrix}0 & 1 & 2\\ 1 & -1 & 4\\ 5 & 1 & 6\end{pmatrix}$ and $B=\begin{pmatrix}1 & 0 & 1\\ 1 & 0 & 0\\ 0 & 0 & 2\end{pmatrix}$. Then find

   a. $3A$ b. $-\frac{1}{2}B$ c. $3A+2B$

   d. matrix C such that $A-2C=B$ e. $A-5B$ f. $3A-2B$

3. Let $A=\begin{pmatrix}1 & -2 & 3\\ 0 & -5 & 2\end{pmatrix}$, $B=\begin{pmatrix}-3 & 2 & 1\\ 4 & 0 & -2\end{pmatrix}$ and $C=\begin{pmatrix}2 & 2 & 0\\ 8 & 0 & -6\end{pmatrix}$.

   Then, determine $2A+(-3)B+\frac{1}{2}C$.

3.2.5 Multiplication of Matrices

Activity 3.8

Consider the following two matrices A and B, where A shows the type and quantity of certain types of food items that Paulos (P) and Meti (M) want to buy and matrix B shows the cost of the items in two supermarkets X and Y.

$$A = \begin{array}{c} \\ P \\ M \end{array}\begin{array}{c}\begin{matrix} C & L & Po\end{matrix}\\ \begin{pmatrix}2 & 4 & 5\\ 1 & 7 & 3\end{pmatrix}\end{array} \qquad B = \begin{array}{c}\begin{matrix} X & Y\end{matrix}\\ \begin{pmatrix}120 & 110\\ 55 & 60\\ 35 & 30\end{pmatrix}\end{array}\begin{array}{c} \\ C \\ L \\ Po\end{array}$$

where, C = Cereal Packets, L = Loaves of Bread and Po = Potatoes (Kg)

a. Calculate the shopping bill for items in each supermarket.

b. Where should they buy, in terms of shopping bill, at X or Y?

In Activity 3.8, to determine the shopping bill for items in each supermarket, you have to add the products of the respective entries of rows of matrix A with the columns of matrix B.

Using this idea, multiplications of matrices are defined as follows.

Definition 3.6

Let $A = (a_{ij})_{m\times p}$ and $B = (b_{kl})_{p\times n}$ be matrices in $\mathbb{R}$, where m, n and p are positive integers. The product of A and B, denoted by AB, is the $m \times n$ matrix $AB = (c_{ij})_{m\times n}$, where, $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots .. + a_{ip}b_{pj}$, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

That is, the ij^{th} entry of the product AB is the sum of the product of the corresponding entries of the i^{th} row of A with the j^{th} column of B.

1. If $A = (a \quad b)$ and $B = \begin{pmatrix} c \\ d \end{pmatrix}$, then $AB = ac + bd$, a real number.
2. If $A = \begin{pmatrix} a \\ b \end{pmatrix}$ and $B = (c \quad d)$, then AB $= \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}$.
3. If A $= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and B $= \begin{pmatrix} d & e \\ f & g \end{pmatrix}$, then AB $= \begin{pmatrix} ad + bf & ae + bg \\ cd + df & ce + dg \end{pmatrix}$

Observe that, if A is an $m \times n$ matrix and B is an $s \times t$ matrix, then AB is defined only when $n = s$ and the product will be an $m \times t$ matrix.

That is, the product AB is defined only when the number of columns of the first matrix,A, is equal to the number of rows of the second matrix, B.

In this case, we say that the two matrices are compatible for matrix multiplication.

Example 1

Determine AB for each of the following:

a. A $= (1 \quad 2)$ and B $= \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

b. A $= \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and B $= (3 \quad 4)$.

c. A $= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and B $= \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}$

d. A $= \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$ and B $= \begin{pmatrix} 0 & 1 & -2 \\ 4 & 9 & 3 \end{pmatrix}$.

Solution

a. A is a 1×2 matrix and B is a 2×1 matrix, then the product AB is a 1×1 matrix or a real number.

That is, $AB = (1 \quad 2)\begin{pmatrix}3\\4\end{pmatrix} = (1 \times 3) + (2 \times 4) = 3 + 8 = 11.$

b. A is a 2×1 matrix and B is a 1×2 matrix, then the product AB is a 2×2. That is $AB = \begin{pmatrix}1\\2\end{pmatrix}(3 \quad 4) = \begin{pmatrix}1 \times 3 & 1 \times 4\\2 \times 3 & 2 \times 4\end{pmatrix} = \begin{pmatrix}3 & 4\\6 & 8\end{pmatrix}.$

c. A is a 2×2 matrix and B is a 2×2 matrix, then the product AB is a 2×2.

That is $AB = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\begin{pmatrix}5 & 7\\6 & 8\end{pmatrix} = \begin{pmatrix}1(5)+2(6) & 1(7)+2(8)\\3(5)+4(6) & 3(7)+4(8)\end{pmatrix} = \begin{pmatrix}17 & 23\\39 & 53\end{pmatrix}.$

d. A is a 2×2 matrix and B is a 2×3 matrix, then the product AB is defined and it is a 2×3 matrix, given by

$$AB = \begin{pmatrix}1(0)+3(4) & 1(1)+3(9) & 1(-2)+3(3)\\2(0)+7(4) & 2(1)+7(9) & 2(-2)+7(3)\end{pmatrix} = \begin{pmatrix}12 & 28 & 7\\28 & 65 & 17\end{pmatrix}.$$

Note that, for matrices A and B in (d) of Example 1, the product BA is not defined, because B is a 2×3 matrix and A is 2×2 matrix.

That is, the number of columns of B is not equal to the number of rows of A.

Example 2

Consider the following two matrices $A = \begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$ and $B = \begin{pmatrix}1 & 1\\2 & 2\end{pmatrix}$.

Then, $AB=\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\2 & 2\end{pmatrix} = \begin{pmatrix}3 & 3\\3 & 3\end{pmatrix}$ and $BA = \begin{pmatrix}1 & 1\\2 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} = \begin{pmatrix}2 & 2\\4 & 4\end{pmatrix}.$

This implies both AB and BA are defined, but $AB \neq BA$.

The example illustrates that, in general, multiplication of matrices is not commutative.

Example 3

Let $A = (a_{ij})_{3\times 2}$ and B $= (b_{ij})_{m\times n}$. Find numbers m and n such that

a. AB is defined;

b. BA is defined;

c. both AB and BA are defined.

Solution

It is given that A is a 3 by 2 matrix.

a. the product AB is defined, if $m = 2$ and n can be any positive integers;

b. the product BA is defined, if $n = 3$ and m can be any positive integers;

c. both products AB and BA are defined only when $m = 2$ and $n = 3$.

Exercise 3.6

Compute the product AB for each of the following cases.

a. $A = \begin{pmatrix} 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

b. $A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \end{pmatrix}$

c. $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 3 & 7 \end{pmatrix}$

d. $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 3 & 1 \end{pmatrix}$

Properties of Multiplication of Matrices

Definition 3.7

Let A $= (a_{ij})_{m\times n}$ be a matrix. Then A is said to be a square matrix, if the number of rows and the number columns of A are equal, $m = n$. In this case, A is called a square matrix of order n.

Example 4

a. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a square matrix of order 2.

b. $B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is a square matrix of order 3.

Activity 3.9

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Then, compute

a. AI_2 and I_2A

b. BI_3 and I_3B

In Activity 3.9, you have observed that $AI_2 = A = I_2A$ and $BI_3 = B = I_3B$. That is,

a. I_2 is the identity matrix for multiplication of matrices of order 2;

b. I_3 is the identity matrix for multiplication of matrices of order 3.

Next, we define the identity matrix for matrix multiplication.

Definition 3.8

The square matrix order n of defined by, $I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{n \times n}$, where 1 on the main diagonal and 0 other wise is called identity matrix for multiplication.

That is, for any square matrix A of order n, $AI_n = A = I_nA$.

The identity matrix of order 2 is given by $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the identity matrix of order 3 is $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and so on .

Activity 3.10

Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. Then compute

a. A(B + C) and AB+AC.

b. (B+C)A and BA+CA

c. A(BC) and (AB)C

d. 2(AB), A(2B) and (2A)B.

In your responses in Activity 3.10, you have observed the following properties of matrix multiplication:

a. A(B + C) = AB+AC and (B+C)A=BA+CA , (multiplication of matrices is distributive over addition of matrices);

b. A(BC) = (AB)C, (matrix multiplication of matrices is associative);

c. 2(AB) = A(2B)=(2A)B.

These properties are summarized as follows.

Properties of Matrix Multiplication

Let A, B and C be matrices in ℝ and $k \in \mathbb{R}$. Assume that the indicated operations on matrices are defined. Then

a. A(B + C) = AB + BC (Multiplication of matrices is left distributive over addition).

b. (A + B)C = AC + BC (Multiplication of matrices is right distributive over addition).

c. A(BC) = (AB)C (Multiplication of matrices is associative).

d. $k(AB) = (kA)B = A(kB)$

e. If 0 is the zero matrix of appropriate size, then 0A = 0 and A0 = 0.

From your knowledge in the set of real numbers, recall that:

(i) if a and b are real numbers such that $ab = 0$, then either $a = 0$ or $b = 0$ (zero product rule);

(ii) if a, b and c are real numbers such that $ab = ac$ and $a \neq 0$, then $b = c$.

That is, cancellation of multiplication works on the set of real numbers.

However, these properties are not true for multiplications of matrices in general as it can be seen in the following examples.

Example 5

If $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}$, then find AB.

Solution

$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and observe that neither A nor B are the zero matrix.

Example 6

If $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} -1 & -1 \\ 3 & 4 \end{pmatrix}$, then find both AB and AC.

Solution

$AB = \begin{pmatrix} 5 & 7 \\ 10 & 14 \end{pmatrix}$ and $AC = \begin{pmatrix} 5 & 7 \\ 10 & 14 \end{pmatrix}$.

Then observe that, $AB = AC$, but $B \neq C$.

Exercise 3.7

1. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$. Then find:

   a. AB b. BA c. (AB)C

   d. A(BC) e. I_2A f. BI_2

2. Let A be a 2 × 3 matrix and B be an $m \times n$ matrix. Find m and n so that:

   a. AB is defined b. BA is defined c. both AB and BA are defined

3. Let $A = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}$. Then determine AB.

4. Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 4 & 0 \\ -1 & 1 \end{pmatrix}$. Determine AB and AC.

3.2.6 Transpose of a Matrix and Its Properties

Activity 3.11

Let A= $\begin{pmatrix} 1 & 2 & 3 \\ -4 & 3 & 0 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & -4 \\ 2 & 3 \\ 3 & 0 \end{pmatrix}$. Can you mention how B is obtained from A?

From your responses in Activity 3.11, you have observed that the rows of A are columns of B and the columns of A are rows of B.

That is, B is obtained from A by changing rows of A to columns of B and columns of A to rows of B and this process is called transposing a matrix.

Definition 3.9

Let A = $(a_{ij})_{m\times n}$ be an $m \times n$ matrix. Then the matrix B = $(b_{ji})_{n\times m}$ where $b_{ji} = a_{ij}$ for $1 \leq j \leq n$ and $1 \leq i \leq m$ is called the transpose of A and denoted by A^t.

That is, if A = $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$, then A^t = $\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$.

Example 1

Find the transpose of each of the following matrices.

a. A = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

b. B = $\begin{pmatrix} 1 & -4 \\ 3 & 0 \\ 2 & 1 \end{pmatrix}$

Solution

a. $A^t = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

b. $B^t = \begin{pmatrix} 1 & -4 \\ 3 & 0 \\ 2 & 1 \end{pmatrix}^t = \begin{pmatrix} 1 & 3 & 2 \\ -4 & 0 & 1 \end{pmatrix}$.

Activity 3.12

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix}$. Then find,

a. $(A + B)^t$ and $A^t + B^t$ **b.** $(3A)^t$ and $3A^t$

c. $(AB)^t$ and B^tA^t **d.** $(A^t)^t$

From your response in Activity 3.12, observe that;

a. $(A + B)^t = A^t + B^t$ b. $(3A)^t = 3A^t$

c. $(AB)^t = B^tA^t$ d. $(A^t)^t = A$.

These properties of transposes of matrices are true in general and they are given as follows.

Properties of Transposes of Matrices

Let A and B be matrices such that the given operations are defined and $k \in \mathbb{R}$. Then

a. $(A + B)^t = A^t + B^t$ b. $(kA)^t = kA^t$ c. $(AB)^t = B^tA^t$ d. $(A^t)^t = A$

Example 2

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. Then, compute $(A + B)^t$, $5A^t$, $(AB)^t$ and $(B^t)^t$

Solution

$A^t = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ and $B^t = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$. Then,

a. $(A+B)^t = A^t + B^t = \begin{pmatrix} 1+0 & 3+1 \\ 2+2 & 1+0 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$.

b. $5A^t = \begin{pmatrix} 5\times 1 & 5\times 3 \\ 5\times 2 & 5\times 1 \end{pmatrix} = \begin{pmatrix} 5 & 15 \\ 10 & 5 \end{pmatrix}$.

c. $(AB)^t = B^tA^t = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} (0\times 1)+(1\times 2) & (0\times 3)+(1\times 1) \\ (2\times 1)+(0\times 2) & (2\times 3)+(0\times 1) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 6 \end{pmatrix}$

d. $(B^t)^t = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}^t = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = B$.

Exercise 3.8

1. Let $A = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$ Then find

a. A^t b. B^t c. $(A+B)^t$

d. $(AB)^t$ e. $(B^t)^t$ f. $(4AB)^t$

2. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Then find

a. A^t b. $(A+B)^t$ c. B^t

d. $(AB)^t$ e. $(B^t)^t$ f. $(2AB)^t$

3.3 Special Types of Matrices

3.3.1 Symmetric and Skew Symmetric Matrices

Activity 3.13

Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$. Find

a. A^t b. B^t c. $B + B^t$

From your responses in Activity 3.13, observe that

a. $A^t = A$ b. $B^t = -B$ c. $B + B^t = 0$

Definition 3.10

Let $A = (a_{ij})_{n\times n}$ be a square matrix of order n. Then A is said to be

a. a symmetric matrix, if $A = A^t$.

That is, A is a symmetric matrix, if $a_{ij} = a_{ji}$ for $1 \le i, j \le n$.

b. a skew-symmetric matrix, if $A = -A^t$.

That is, A is a skew symmetric matrix if $A + A^t = 0_{n\times n}$.

Equivalently A is skew symmetric matrix if $a_{ij} = -a_{ji}$ for $1 \le i, j \le n$.

Note that, if matrix $A = (a_{ij})_{n\times n}$ is skew symmetric matrix, then $a_{ii} = 0$ for $i = 1, 2, \dots, n$.

Example 1

Identify the following matrices as symmetric, skew-symmetric or neither.

I_n, for a positive integer n, $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & -4 \\ -3 & 4 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 & 3 \\ 5 & 0 & 7 \\ -3 & 2 & 4 \end{pmatrix}$.

Solution

a. $I_n^t = I_n$ and hence I_n is a symmetric matrix.

b. $A^t = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{pmatrix} = A$, hence A is a symmetric matrix.

c. $B^t = \begin{pmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{pmatrix}$ and $B^t = -B$. This implies B is a skew-symmetric matrix.

d. $C^t = \begin{pmatrix} 1 & 5 & -3 \\ 0 & 0 & 2 \\ 3 & 7 & 4 \end{pmatrix}$. This implies $C \neq C^t$ and $C \neq -C^t$, hence C is neither a symmetric matrix nor a skew-symmetric matrix.

Example 2

Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ -4 & 3 & 8 \\ 6 & -10 & 5 \end{pmatrix}$. Then find

a. $B = \frac{1}{2}(A + A^t)$ b. $C = \frac{1}{2}(A - A^t)$ c. $B + C$

Solution

The transpose of A is $A^t = \begin{pmatrix} 1 & -4 & 6 \\ 2 & 3 & -10 \\ 3 & 8 & 5 \end{pmatrix}$

Thus, $A + A^t = \begin{pmatrix} 2 & -2 & 9 \\ -2 & 6 & -2 \\ 9 & -2 & 10 \end{pmatrix}$ and $A - A^t = \begin{pmatrix} 0 & 6 & -3 \\ -6 & 0 & 18 \\ 3 & -18 & 0 \end{pmatrix}$

Then,

a. matrix $B = \frac{1}{2}(A + A^t) = \frac{1}{2}\begin{pmatrix} 2 & -2 & 9 \\ -2 & 6 & -2 \\ 9 & -2 & 10 \end{pmatrix} = \begin{pmatrix} 1 & -1 & \frac{9}{2} \\ -1 & 3 & -1 \\ \frac{9}{2} & -1 & 5 \end{pmatrix}$ is a symmetric matrix;

b. matrix $C = \frac{1}{2}(A - A^t) = \frac{1}{2}\begin{pmatrix} 0 & 6 & -3 \\ -6 & 0 & 18 \\ 3 & -18 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3 & -\frac{3}{2} \\ -3 & 0 & 9 \\ \frac{3}{2} & -9 & 0 \end{pmatrix}$ is a skew-symmetric matrix.

Observe that $B + C = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t) = A$ hence A is written as sum of a symmetric matrix and a skew-symmetric matrix.

Exercise 3.9

1. Identify each of the following matrices as symmetric, skew-symmetric or neither

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 5 \end{pmatrix},\ B = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{pmatrix},\ C = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{pmatrix},\ D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

2. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 1 & 5 \end{pmatrix}$. Find a symmetric matrix B and a skew-symmetric matrix C so that $A = B + C$

3. Let $A = \begin{pmatrix} 1 & a & 2 \\ 0 & 1 & 0 \\ b & c & 5 \end{pmatrix}$. Find the values a, b, c so that A is a symmetric

3.3.2 Diagonal Matrices and Triangular Matrices

Definition 3.11

Let $A = (a_{ij})_{n\times n}$ be a square matrix of order n.

a. The elements $a_{11}, a_{22}, \dots, a_{nn}$ are called diagonal elements of A.

b. The sum of the diagonal elements of A is called the **trace** of A and denoted by tr(A). That is, $\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$

c. A is called a diagonal matrix if $a_{ij} = 0$ for all $i \neq j$. That is, A is a diagonal matrix, if

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

d. A is called a scalar matrix if A is a diagonal matrix and $a_{ii} = \alpha$ for some $\alpha \in \mathbb{R}$ and for $i = 1, 2, \dots, n$. That is, A is scalar matrix, if

$$A = \begin{pmatrix} \alpha & 0 & \dots & 0 \\ 0 & \alpha & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha \end{pmatrix}_{n\times n} \quad \text{for some } \alpha \in \mathbb{R}.$$

Definition 3.12

Let $A = (a_{ij})_{n\times n}$ be a square matrix of order n.

a. If $a_{ij} = 0$ for all $i > j$, then A is called an upper triangular matrix. That is,

A is said to be an upper triangular matrix if $A = \begin{pmatrix} a_{11} & a_{12} & \dots a_{1n} \\ 0 & a_{22} & \dots a_{2n} \\ \vdots & \vdots & \ddots \vdots \\ 0 & 0 & \dots a_{nn} \end{pmatrix}$;

b. A is called a strictly upper triangular matrix if $a_{ij} = 0$ for all $i \geq j$. That is, A is said to be a strictly upper triangular matrix if

$$A = \begin{pmatrix} 0 & a_{12} & \dots a_{1(n-1)} & a_{1n} \\ 0 & 0 & \dots a_{2(n-1)} & a_{2n} \\ \vdots & \vdots & \ddots \vdots & \vdots \\ 0 & 0 & \dots 0 & 0 \end{pmatrix};$$

c. A is called a lower triangular matrix if $a_{ij} = 0$ for all $i < j$. That is A is said to be a lower triangular matrix if

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 & 0 \\ a_{21} & a_{22} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n(n-1)} & a_{nn} \end{pmatrix};$$

d. A is called a strictly lower triangular matrix if $a_{ij} = 0$ for all $i \leq j$ That is A is said to be a strictly lower triangular matrix if

$$A = \begin{pmatrix} 0 & 0 & \dots 0 & 0 \\ a_{21} & 0 & \dots 0 & 0 \\ \vdots & \vdots & \ddots \vdots & \vdots \\ a_{n1} & a_{n2} & \dots a_{n(n-1)} & 0 \end{pmatrix}.$$

Example

Identify each of the following matrices as diagonal, scalar or triangular matrices.

a. $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -6 \\ 0 & 0 & 5 \end{pmatrix}$ b. $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 2 & 0 \end{pmatrix}$ c. $C = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

Solution

a. $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -6 \\ 0 & 0 & 5 \end{pmatrix}$ is an upper triangular matrix, because all the entries below the main diagonal are 0.

b. $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 2 & 0 \end{pmatrix}$ is a strictly lower triangular matrix, because all the entries below the main diagonal and on the main diagonal are 0.

c. $C = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ is both a scalar and a diagonal matrix, because all the entries below and above the main diagonals are 0 and the entries in the main diagonal are all -2.

Exercise 3.10

1. Identify each of the following matrices as diagonal, scalar or triangular matrices.

$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 2 & -3 & -3 \end{pmatrix}$, $C = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$.

2. Let $A = \begin{pmatrix} 1 & a & 2 \\ 0 & 1 & 0 \\ b & c & 5 \end{pmatrix}$. Find a, b, c such that A is an upper triangular matrix.

3.4 Elementary Row Operations of Matrices

Activity 3.14

Consider the system

$$x + y = 1$$
$$x - 2y = -11$$

a. Multiply the first equation by 2 and add the result to the second equation.

b. Multiply the result in (a) by $\frac{1}{3}$ and solve for x.

The operations you have performed in Activity 3.14 are operations on the rows of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & -11 \end{pmatrix}$ of the given system.

A familiar method of solving a set of linear equations is by elimination of the unknowns. We shall systematize this method by working only on the numbers involved in the given set of linear equations.

Definition 3.13

Let m and n be positive integers and A be an $m \times n$ matrix with entries in $\mathbb{R}$. An elementary row operations on A is any one of the following operation.

Swapping: Interchanging two rows of matrix A:

Swapping of the i^{th} row and the j^{th} row of A is denoted by $R_i \leftrightarrow R_j$.

Scaling: Multiplying a row of A by a non-zero constant:

Multiplying the i^{th} row by a nonzero scalar k is denoted by $R_i \rightarrow kR_i$.

Pivoting: Adding a nonzero constant multiple of one row of A onto another row of A.

Adding k times the i^{th} row of A onto the j^{th} row is denoted by $R_j \rightarrow R_j + kR_i$

Example 1

Let $A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$. Apply each of the following elementary row operations one after the other on A and obtain a new matrix.

a. $R_1 \leftrightarrow R_2$ b. $R_2 \to R_2 - 4R_1$ c. $R_2 \to -\frac{1}{10}R_2$

Solution

a. $A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ (Interchanging R_1 by R_2).

b. $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 4R_1} \begin{pmatrix} 1 & 3 \\ 0 & -10 \end{pmatrix}$ (Replacing R_2 by $R_2 - 4R_1$)

c. $\begin{pmatrix} 1 & 3 \\ 0 & -10 \end{pmatrix} \xrightarrow{R_2 \to -\frac{1}{10}R_2} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ (Multiplying R_2 by $-\frac{1}{10}$)

Thus, $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ is an upper triangular matrix.

Example 2

Let $A = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 2 & 7 & 12 \end{pmatrix}$. Apply appropriate elementary row operations to change A into an upper triangular matrix.

Solution

$A = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 2 & 7 & 12 \end{pmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 7 & 12 \end{pmatrix}$ (Multiplying R_1 by $\frac{1}{2}$)

$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 4R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 3 & 6 \end{pmatrix}$ (Replacing R_2 by $R_2 - 4R_1$ and R_3 by $R_3 - 2R_1$)

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix} \text{(Replacing } R_3 \text{by } R_3 + R_2\text{)}$$

The matrix $\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$ is an upper triangular matrix obtained from A.

Definition 3.14

The inverse operation of an elementary row operation is an elementary row operation of the same type. That is,

a. The inverse of the elementary row operation $R_i \leftrightarrow R_j$ is $R_j \leftrightarrow R_i$, which is again an elementary row operation.

b. For a nonzero real number k, the inverse of the elementary row operation $R_j \longrightarrow kR_j$ is $R_j \longrightarrow \frac{1}{k}R_j$ which is again an elementary row operation.

c. For a scalar k the inverse of the elementary row operation $R_i \longrightarrow R_i + kR_j$ is $R_i \longrightarrow R_i - kR_j$ which is also an elementary row operation.

Example 3

Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix}$. Then, find the inverse of the following elementary row operations:

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 5 \\ 1 & 2 \end{pmatrix} \text{ (Interchanging } R_1 \text{ and } R_2\text{)}$$

$$\xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 5 \\ 0 & -3 \end{pmatrix} \text{ (Replacing } R_2 \text{ by } R_2 - R_1\text{)}$$

$$\xrightarrow{R_2 \to -\frac{1}{3}R_2} \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \text{(Replacing } R_2 \text{by} -\frac{1}{3}R_2\text{)}$$

Solution

Let us apply the inverse elementary row operations of the above three elementary row operations on the matrix $\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$ as follows.

$\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_2 \to -3R_2} \begin{pmatrix} 1 & 5 \\ 0 & -3 \end{pmatrix}$ (Multiplying R_2 by -3, which is the inverse of the elementary row operation , multiplying $R_2$by $-\frac{1}{3}$).

$\xrightarrow{R_2 \to R_2 + R_1} \begin{pmatrix} 1 & 5 \\ 1 & 2 \end{pmatrix}$ (Replacing $R_2$by $\mathrm{R}_2 + \mathrm{R}_1$, which is the inverse of the elementary row operation, replacing $R_2$by $R_2 - R_1$).

$\xrightarrow{R_2 \leftrightarrow R_1} \begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix}$ (Interchanging $R_2$and R_1)

In Example 3, observe that

a. $R_2 \longrightarrow -3R_2$ is the inverse of $R_2 \longrightarrow -\frac{1}{3}R_2$

b. $R_2 \longrightarrow R_2 + R_1$ is the inverse of $R_2 \longrightarrow R_2 - R_1$ and

c. $R_2 \leftrightarrow R_1$ is the inverse of $R_1 \leftrightarrow R_2$

Elementary Column Operations

There are also column operations on matrices.

Swapping: Interchanging two columns of a matrix.

Scaling: Multiplying a column of a matrix by a non-zero constant.

Pivoting: Adding a constant multiple of one column of a matrix onto another column.

However, in this textbook, we use only elementary row operations in solving system of linear equations.

Exercise 3.11

1. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 6 & -5 \end{pmatrix}$. Find the matrix obtained from A by applying the following elementary row operation.

   a. $R_1 \leftrightarrow R_3$ b. $R_2 \to R_2 - 4R_1$ c. $R_3 \to \frac{1}{3}R_3$

2. Let $B = \begin{pmatrix} 1 & 1 & 2 \\ -2 & 1 & -5 \\ 2 & 3 & -1 \end{pmatrix}$. Determine the inverse of each of the following elementary row operations on B.

   a. $R_2 \leftrightarrow R_3$ b. $R_3 \to R_3 + 2R_1$ c. $R_2 \to 4R_2$

Row Echelon Form (REF) of a Matrix

Definition 3.15

Given two matrices A and B of the same size, A and B are said to be row-equivalent if one of the matrices is obtained from the other by successive applications of a finite number of elementary row operations and this relation is denoted by $A \cong B$.

Example 4

Let $A = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$. Find three matrices that are row equivalent to A.

Solution

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \text{ (Interchanging } R_1 \text{ and } R_2\text{)}$$

$$\xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix} \text{ (Replacing } R_2 \text{ by } R_2 - R_1\text{)}$$

$$\xrightarrow{R_2 \to -\frac{1}{3}R_2} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ (Replacing } R_2 \text{ by } -\frac{1}{3}R_2\text{)}$$

Then the matrices $\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are all row equivalent to matrix A.

Note

Row equivalence of matrices helps us to transform a given matrix to a matrix with greater number of zeros than the original matrix and this simplifies operations on matrices.

Definition 3.16 (Row Echelon Form (REF))

An $m \times n$ matrix A is said to be in it **Row Echelon Form (REF)** if it satisfies the following.

(a) All rows consisting entirely zeros, if any, are at the bottom of the matrix.

(b) If R_i and $R_{(i+1)}$ are any two successive nonzero rows of A, the number of zeros preceding the leading element of the $(i+1)^{th}$ row is greater than the number of such zeros in i^{th} row.

Note

By a zero row we mean that a row in which all the entries of the row are zero.

The matrix $A = \begin{pmatrix} 1 & a & b & c & d \\ 0 & 0 & 2 & e & f \\ 0 & 0 & 0 & 0 & g \end{pmatrix}$ is in Row Echelon Form.

Example 5

Determine if each of the following matrices are in Row Echelon Form.

$$A=\begin{pmatrix}1&2&2\\0&1&1\\0&0&1\end{pmatrix}, B=\begin{pmatrix}1&-1&2\\0&1&0\\0&0&1\\0&0&0\end{pmatrix}, C=\begin{pmatrix}1&1&-1&0\\0&0&0&0\\0&0&1&3\\0&0&0&1\end{pmatrix},$$

$$D=\begin{pmatrix}1&1&0&4\\0&1&0&0\\0&0&1&3\\0&0&1&1\end{pmatrix} \text{ and } E=\begin{pmatrix}1&2&1&-2\\0&2&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

Solution

a. Matrices A, B and E are in Row Echelon Form.

b. Matrix C is not in Row Echelon Form, because C has a zero row, the second row, but this zero row is not at the bottom as the third row is not a zero row.

c. Matrix D is not in row echelon form, because the number of zeros preceding the leading element of the third and the fourth row are both two.

Exercise 3.12

Determine if each of the following matrices in **Row Echelon Form**.

a. $\begin{pmatrix}1&2&3\\0&0&6\\1&1&0\end{pmatrix}$ b. $\begin{pmatrix}1&2&1&3\\0&4&5&0\\0&0&0&1\end{pmatrix}$ c. $\begin{pmatrix}1&2&1&2\\0&0&0&3\\0&1&1&4\end{pmatrix}$

Reduced Row Echelon Form (RREF) of a Matrix

Activity 3.15

Consider the matrices $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 2 & 1 \end{pmatrix}$.

a. Give your reason why matrices A and B are not in Row Echelon Form

b. Apply the elementary row operation $R_2 \leftrightarrow R_3$ on A and obtain matrix C.

c. Apply the elementary row operations $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_2$ respectively on matrix B and obtain matrix D.

d. What is the relationship between matrices C and D in (b) and (c) above respectively?

The matrices you have obtained in your responses in *Activity 3.15* are matrices in Row Echelon Form. Thus, any matrix is row equivalent to a matrix in Row Echelon Form.

Example 6

Reduce the matrix $A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 2 & 1 & 1 & 2 \\ -1 & 0 & -2 & -4 \end{pmatrix}$ into a matrix in Row Echelon Form.

Solution

A is not in REF, because the number of zeros preceding the first nonzero element in the second row is zero which is the same as the number of zeros preceding the first nonzero element in the first row.

Therefore, to reduce A into a matrix in REF, choose appropriate elementary row operations.

$$A=\begin{pmatrix}1&3&0&2\\2&1&1&2\\-1&0&-2&-4\end{pmatrix}\xrightarrow[R_3\to R_3+R_1]{R_2\to R_2-2R_2}\begin{pmatrix}1&3&0&2\\0&-5&1&-2\\0&3&-2&-2\end{pmatrix}$$

$$\xrightarrow{R_3\to R_3+\frac{3}{5}R_2}\begin{pmatrix}1&3&0&2\\0&-5&1&-2\\0&0&-2&-\frac{4}{5}\end{pmatrix}=B.$$

The last matrix, B is in Row Echelon Form (REF), obtained from A by applying the three successive elementary row operations, $R_2\to R_2-2R_2$, $R_3\to R_3+R_1$ and $R_3\to R_3+\frac{3}{5}R_2$.

Definition 3.17

Given positive integers m and n, an $m\times n$ matrix A is said to be in **Reduced Row Echelon Form (RREF)** if A is in **Row Echelon Form (REF)** and the leading element in each nonzero row is 1 and it is the only nonzero number in its column.

Example 7

Determine which of the following matrices are in Reduced Row Echelon Form.

$$I_3=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad A=\begin{pmatrix}1&0&0\\0&1&2\\0&0&1\end{pmatrix},\quad B=\begin{pmatrix}1&0&2&5\\0&1&2&6\\0&0&0&0\end{pmatrix},$$

$$C=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad D=\begin{pmatrix}1&0&0\\0&0&1\\0&0&0\\0&0&0\end{pmatrix}.$$

Solution

The matrices I_3, B and D are in RREF, because all the three matrices satisfy the requirements of the definition of a matrix in RREF.

(a) Matrix A is not in RREF, because the leading element of the third row of A, which is 1, is not the only nonzero element in its column.

(b) Matrix C is not in RREF, because the number of zeros preceding the leading element of the second row is two and the number of zeros preceding the leading element of the third row is one.

However, you can reduce both A and C in the above example to RREF by applying appropriate elementary operations.

Activity 3.16

Consider the matrices $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

a. Apply the elementary row operation $R_2 \to R_2 - 2R_3$ on matrix A.

b. Apply the elementary row operation $R_2 \leftrightarrow R_3$ on matrix C.

In your responses for (a) and (b) of Activity 3.16, observe that the matrices obtained after the proposed elementary operations are in Reduced Row Echelon Form (RREF).

Activity 3.16 also illustrates that you can use proper elementary operations to reduce a matrix into a matrix in Reduced Row Echelon Form (RREF).

Remark

Every $m \times n$ matrix is row equivalent to an $m \times n$ matrix in Reduced Row Echelon Form (RREF).

Example 8

Reduce the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ into a matrix in Reduced Row Echelon Form.

Solution

A is not in RREF, because

a. the first nonzero element in the third row is -2, not 1;

b. the first nonzero element of the third row is not the only nonzero element in its column.

Therefore, to reduce A into a matrix in RREF, choose appropriate elementary row operations.

$$A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_3 \to -\frac{1}{2}R_3} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 \to R_2 - R_3} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = B.$$

The last matrix, B, obtained from A by applying the two successive elementary row operations, $R_3 \to -\frac{1}{2}R_3$ and $R_2 \to R_2 - R_3$ is in RREF.

Exercise 3.13

1. Reduce each of the following matrices into **Row Echelon Form**.

a. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & 7 \end{pmatrix}$ b. $\begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 5 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ c. $\begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 1 & 3 \\ 1 & 4 & 1 & 4 \end{pmatrix}$

2. Reduce each of the following matrices into Reduced Row Echelon Form.

a. $\begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{pmatrix}$ b. $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix}$ c. $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 3 \end{pmatrix}$

3.5 Systems of Linear Equations with Two or Three Variables

One of the main reasons of studying matrices is to solve systems of linear equations. In this section, you will learn techniques or methods of solving systems of linear equations.

Activity 3.17

The sum of two natural numbers is 30 and the second number is 4 more than the first number. Find the numbers.

In your responses in Activity 3.17, if you let the first number by x and these second number by y, then you need to find a common solution to the two solutions $x + y = 30$ and $y = x + 4$.

The two equations $x + y = 30$ and $y = x + 4$ are called linear equations and the two equations together form a set of linear equations called a system of linear equations.

Definition 3.18

An equation of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, where $a_1, a_2, \cdots, a_n, b$ are real numbers, is called a linear equation in the$n$ variables $x_1, x_2, \cdots, x_n$. The variables $x_1, x_2, \cdots, x_n$ are also called unknowns.

Example 1

a. The equation $2x - y = 5$ is a linear equation in two variables, x and y .

b. The equation $2x + 3y + 4z = 5$ is a linear equation in three variables, x, y and z.

c. The equation $x_1 + \pi x_2 + 4x_3 + x_4 + \frac{1}{3}x_5 = 0$ is a linear equation in five variables, x_1, x_2, x_3, $x_4$and x_5.

Remark

A linear equation does not involve product of variables, quotient of variables or roots of variables and all the variables that occur in the equation are only to power one.

Example 2

Give your reasons why each of the following equations is not a linear equation. $3x^2 + 5y = 7,\ xy - 2z + w = 1$ and $x - \sqrt{y} + 4z = 0$.

Solution

a. The equation $3x^2 + 5y = 7$ is not a linear equation, because it involves x^2.

b. The equation $xy - 2z + w = 1$ in not a linear equation, because it involves xy.

c. The equation $x - \sqrt{y} + 4z = 0$ is not a linear equation, because it involves $\sqrt{y}$.

Activity 3.18

Consider the equation $x - y = 5$.

a. Substitute $x = 2$ and $y = 3$ and determine the result.

b. Substitute $x = 6$ and $y = 1$ and determine the result.

In your responses in Activity 3.18,

a. the ordered pair $(2, 3)$ gives you a false statement;

b. the ordered pair $(6, 1)$ gives you a true statement.

Thus, the ordered pair $(6, 1)$ is a solution to the given equation, whereas the ordered pair $(2, 3)$ is not a solution.

Definition 3.19

An $n-$tuple of numbers $(c_1, c_2, \cdots, c_n)$ is said to be a solution to a linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

if the equation is satisfied when we substitute $x_1 = c_1, x_2 = c_2, \cdots, x_n = c_n$ in the equation, that is, if $a_1c_1 + a_2c_2 + \cdots + a_nc_n = b$.

The set of all the possible solutions of a given linear equation is called the **solution set** or the **general solution set** of the given linear equation.

Example 3

Consider the linear equation $2x + y = 4$.

a. Which of the ordered pairs $(2,0), (0,2)$ and $(1,2)$ are solutions of the given linear equation?

b. Find the solution set of the given linear equation.

Solution

a. For $x = 2$ and $y = 0$, we have $(2 \times 2) + 0 = 4 + 0 = 4$.

This implies, the ordered pair $(2,\ 0)$ is a solution to the given linear equation.

For $x = 0$ and $y = 2$, we have $(2 \times 0) + 2 = 0 + 2 \neq 4$.

This implies, the ordered pair $(0,\ 2)$ is not a solution to the given linear equation.

For $x = 1$ and $y = 2$, you have$(2 \times 1) + 2 = 2 + 2 = 4$.

This implies, the ordered pair $(1,\ 2)$ is a solution to the given linear equation.

b. From the given equation if you solve for y in terms of x gives you

$2x + y = 4 \Rightarrow y = -2x + 4$.

This implies, the solution set of the given linear equation is the set of all ordered pairs of numbers (x, y) such that $2x + y = 4$ and $y = 4 - 2x$.

Therefore, the solution set of the given equation is $\{(x, 4 - 2x): x \in \mathbb{R}\}$

Exercise 3.14

1. Give your reasons why each of the following equations is not a linear equation.

   $x + 5y^2 = 7,\ x + yz + w = 0$ and $x - y + 4\sqrt{z} = 0$.

2. Consider the linear equation $x - 3y = 5$.

   a. Which of the ordered pairs $(1,1)$, $(2, -1)$and $(5,0)$ are solutions of the given linear equation?

   b. Find the solution set of the given linear equation.

Augmented Matrix of a System of Linear Equations

The main objective of this section is to find the set of all common solutions of a finite number of linear equations having the same variables.

Definition 3.20

Let $a_{ij}, b_i \in \mathbb{R}$, for $i = 1,2,\dots,m$ and $j = 1,2,\dots,n$. A finite set of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

is called a system of m linear equations in n variables; $x_1, x_2, \dots, x_n$

Activity 3.19

Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

a. Find AX

b. Find the values of x and y such that $AX = B$.

In Activity 3.19 part (b), to find the values of x and y, you have to find a solution for the system $\begin{matrix} x + 2y = 1 \\ x + 3y = -4 \end{matrix}$, which is a system of two linear equations in two variables. Then, the expression $AX = B$ is called a matrix form of the system.

Remark

Consider the following system of linear equations.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

The system can be written in matrix form as $AX = B$,where

$$A = (a_{ij})_{m\times n}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

The matrix $A = \left(a_{ij}\right)_{m\times n}$ is called the coefficient matrix of the system and the matrix

$$(A|B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

obtained by adjoining B to A is called the **augmented matrix** of the system.

The coeeficient matrix and the augmented matrix of a given system play important roles in determining the solution set of the given system.

Example 4

Find the coefficient and the augmented matrix of the system.

$$\begin{gathered} x + y + z = 8 \\ x + 2y = 5 \\ y + 3z = 9 \end{gathered}$$

Solution

The system can be rewritten in matrix form as:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \\ 9 \end{pmatrix}.$$

The matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ is the coefficient matrix of the system and the matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 1 & 2 & 0 & 5 \\ 0 & 1 & 3 & 9 \end{array}\right)$$

is the augmented matrix of the given system.

Exercise 3.15

Find the coefficient matrix and the augmented matrix of each of the following systems of linear equations.

a. $\begin{aligned} 2x + y &= 5 \\ x - 3y &= 7 \end{aligned}$

b. $\begin{aligned} x + y + z &= 1 \\ 2x - y - z &= 0 \\ 3x + 5y - 9z &= -1 \end{aligned}$

Solution Set of a System of Linear Equations (1)

Definition 3.21

Consider a system AX=B of m linear equations in n variables.

An order n −tuple of numbers $C = (c_1, c_2, \cdots, c_n)^t$ is a solution to the system $AX = B$ if $AC = B$ and the set of all solutions of $AX = B$ is called the solution set of the given system.

Example 5

Solve the following system of linear equations

$$\begin{aligned} x + 2y &= 1 \\ y + 3z &= 0 \end{aligned}.$$

Solution

Solving for y in terms of z from the second equation gives us $y = -3z$.

Substituting $y = -3z$ in the first equation, $x + 2y = 1$, and solving for x in terms of z gives us $x = 1 - 2y = 1 - 2(-3z) = 1 + 6z$.

Therefore, the solution set of the given system is $\{(1+6z, -3z, z) : z \in \mathbb{R}\}$.

If we let $z = 0$, then $x = 1 + 6(0) = 1$ and $y = -3(0) = 0$.

Therefore, (1, 0, 0) is a particular solution to the given system.

Activity 3.20

Consider the system

$$x - y = 1$$
$$-2x + 2y = 0$$

Multiply the first equation by -2, add the two equations and obtain the result.

In your responses to the question in Activity 3.20, you have obtained the expression $0 = 1$, which is false and hence the given system does not have a solution.

Definition 3.22

A system of linear equations that has at least one solution is called a **consistent system** and a system that does not have a solution is called an **inconsistent system**.

Example 6

Determine if the following system of linear equation is consistent.

$$x - 2y = 1$$
$$-x + 3y = 0$$

Solution

Adding the two equations gives us $y = 1$.

Then, from the first equation, $x - 2(1) = 1 \Longrightarrow x = 3$.

This implies (3,1) is a solution to the given system, hence the given system is a consistent system.

Example 7

Determine if the following system of linear equations is consistent.

$$x - y = 1$$
$$-2x + 2y = 0$$

Solution

Multiplying the first equation by 2 and adding the two equations give us $0 = 2$, which is always false.

Therefore, the given system has no solution and hence the system is an inconsistent system.

Exercise 3.16

Determine if each of the following system of linear equations is consistent or inconsistent.

a. $\begin{array}{r} 2x - y = 2 \\ x + y = 1 \end{array}$

b. $\begin{array}{r} -x + 2y = 5 \\ 2x - 4y = -4 \end{array}$

Solution Set of a System of Linear Equations (2)

Activity 3.21

Find the point(s) of intersections of each of the following pairs of lines (if there is any).

a. $l_1: x + y = 2$ and $l_2: x - y = 0$;

b. $s_1: x - y = 2$ and $s_2: -2x + 2y = 6$;

c. $t_1: x - y = 1$ and $t_2: -2x + 2y = -2$.

From your responses in Activity 3.21, observe that,

a. l_1 and l_2 intersect exactly at one point;

b. s_1 and s_2 do not intersect and they are two parallel lines;

c. t_1 and $t_2$are identical lines.

To find out all the possibilities regarding the solutions of systems of linear equations, consider the case when you have a system of two linear equations in two variables.

Consider a general system of two linear equations in two variables

$$a_1x + b_1y = c_1 \quad (a_1, b_1 \text{not both zero})$$
$$a_2x + b_2y = c_2 \quad (a_2, b_2 \text{not both zero})$$

These are equations of two lines, say $l_1: a_1x + b_1y = c_1$ and

$l_2: a_2x + b_2y = c_2$.

There are only three possibilities about the intersection points of these two lines.

a. The lines l_1 and l_2 intersect exactly at one point and this point of intersection is the only solution of the given system of linear equations, see Figure 3.1.

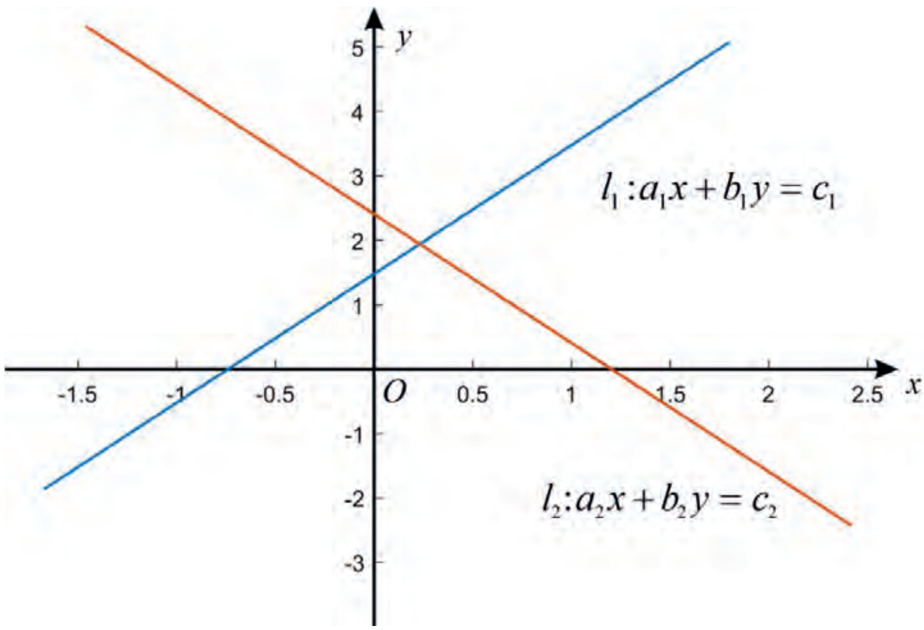


Figure 3.1: Intersecting Lines

b. l_1 and l_2 are two parallel lines and they do not intersect. In this case the given system of linear equations has no solution, see Figure 3.2.

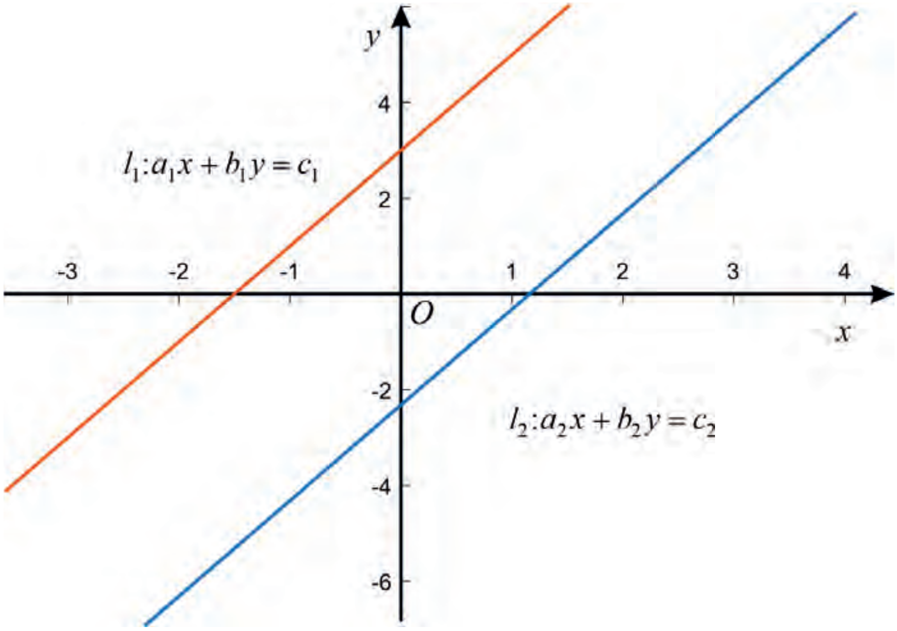


Figure 3.2: Parallel Lines

c. The lines l_1 and l_2 coincide (or identical). In this case, all the ordered pairs of numbers satisfying one of the two equations of the given system on one of the two lines are solutions of the given system and hence the system has infinitely many solutions, see Figure 3.3.

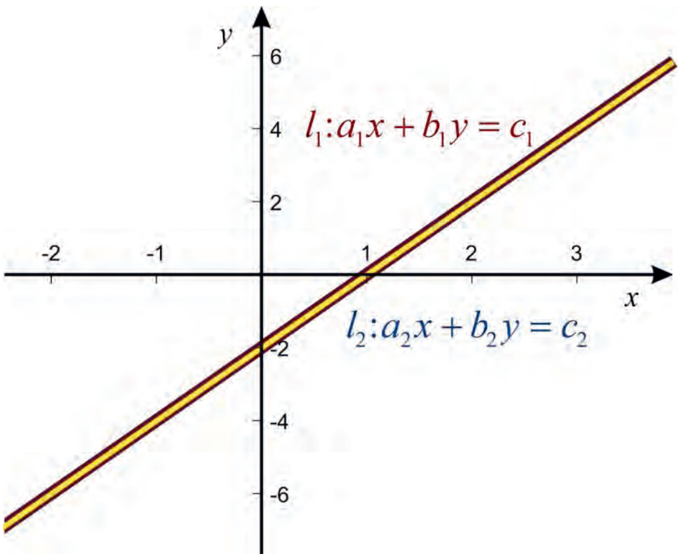


Figure 3.3: Identical Lines

Thus, a given system of linear equations has either **one solution, no solution or infinitely many solutions.**

A consistent system has either only one solution or infinitely many solutions.

Example 8

Determine if each one of the following system of linear equations has only one solution, no solution or infinitely many solutions.

a. $\begin{aligned} x+y&=1 \\ x-y&=3 \end{aligned}$ b. $\begin{aligned} x-y&=0 \\ -2x+2y&=3 \end{aligned}$ c. $\begin{aligned} x+2y&=3 \\ 3x+6y&=9 \end{aligned}$

Solution

a. Adding the two equations gives us $2x = 4 \Longrightarrow x = 2$.

   Then, solving for y from the equation $x+y=1 \Rightarrow y=1-2=-1$.

   Therefore, $(2,-1)$ is the only solution of the given system.

b. Multiplying the first equation by 2 and adding the two equations gives us $0=3$, which is always false. Therefore, the system has no solution.

c. Multiplying the first equation by 3 and subtracting it from the second equation gives us $0 = 0$. This implies, the two equations are identical and hence the system has infinitely many solutions.

Exercise 3.17

Determine if each one of the following system of linear equations has unique solution, no solution or infinitely many solutions.

a. $\begin{aligned} x-y&=1 \\ x+y&=5 \end{aligned}$ b. $\begin{aligned} x+y&=1 \\ 2x+2y&=4 \end{aligned}$ c. $\begin{aligned} x-2y&=1 \\ -2x+4y&=-2 \end{aligned}$

Homogeneous Systems of Linear Equations

Activity 3.22

Consider the system $\begin{matrix} x+y=0 \\ x-2y=0 \end{matrix}$.

a. Write the given system in matrix form, AX=B.

b. What type of matrix is B in (a)?

The matrix B in your responses of Activity 3.22 is the zero matrix and the system is called a homogenous system.

Definition 3.23

Consider a system of linear equations

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{matrix}$$

If $b_1 = b_2 = \cdots = b_m = 0$, then the system is called a homogeneous, otherwise, it is called a non-homogeneous system.

Note

A homogenous system is a system of the form AX = 0.

Example 9

Write the following system in matrix form.

$$\begin{matrix} x + y + z = 0 \\ 2y + z = 0 \\ x + 3z = 0 \end{matrix}$$

Solution

The system in matrix form is $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and the system is homogeneous system.

Remark

Consider a homogeneous system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

a. The given system has always a solution; namely, $x_1 = x_2 = \cdots = x_n = 0$.
   This solution is called the **trivial** solution of the homogenous system.

b. A solution $(x_1, x_2, \cdots, x_n)^t$ of the given system for which $x_i \neq 0$, for some $i = 1,2,\ldots,n$, is called a **non-trivial** solution of the given homogenous system.

Example 10

Determine the coefficient matrix and the augmented matrix of the following system of linear equations.

$$\begin{aligned} x + 2y + z &= 0 \\ y + z &= 0 \\ x + y &= 0 \end{aligned}$$

Solution

The given system in matrix form is

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then the coefficient matrix of the system is $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and the augmented matrix of the system is $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right)$

Exercise 3.18

Find the coefficient matrix and the augmented matrix of each of the following system of linear equations.

a. $\begin{matrix} x + y = 0 \\ 2x - 3y = 0 \end{matrix}$

b. $\begin{matrix} x + y - z = 0 \\ 2x + y + z = 0 \\ x - y - z = 0 \end{matrix}$

c. $\begin{matrix} 2x + 3y - z = 0 \\ x - y + z = 0 \\ x + y - z = 0 \end{matrix}$

3.6 Solutions of Systems of Linear Equations

In this section, you will learn how to solve systems of linear equations that were introduced in Section 3.5, using the concept of elementary row operations.

Activity 3.23

Solve each of the following systems of Linear Equations.

a. $\begin{matrix} x + y = 1 \\ y = 2 \end{matrix}$

b. $\begin{matrix} -x + y = 3 \\ x + y = 1 \end{matrix}$

From your solutions in *Activity 3.23*, observe that the two systems have the same solution and such systems are called equivalent systems..

Definition 3.24

Two systems of linear equations over $\mathbb{R}$, the set of real numbers, with the same number of unknowns are said to be equivalent if every solution of one system is a solution of the other system.

Example 1

Solve the following system of linear equations.

$$x + y = 1$$
$$x - y = 3$$

Solution

The augmented matrix of the system is $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \end{array}\right)$.

Then, reduce this matrix into a matrix in Row Echelon Form using appropriate elementary row operations as follows.

$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \end{array}\right) \xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 2 \end{array}\right)$ and this last matrix is in Row Echelon Form.

Thus, the systems with augmented matrices $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \end{array}\right)$ and $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 2 \end{array}\right)$ have the same solution set.

Then, solve the system

$$x + y = 1$$
$$-2y = 2$$

From the second equation, we get $y = -1$ and substituting this in the first equation gives us that $x = 1 - (-1) = 2$.

Therefore, the solution set of the given system is $\{(2, -1)\}$.

Note

A system of linear equations that is equivalent to a given system AX = B can be obtained by applying appropriate elementary row operations on the augmented matrix (A|B).

Consider a system AX = B.

(a) If the matrices (A|B) and (C|D)are row equivalent augmented matrices of two systems, then the systems AX = B and CX = D have the same solutions.

(b) The method of solving the system AX = B by reducing (A|B) into Row Echelon Form (REF) is called **Gaussian Elimination Method**..

(c) The method of solving the system AX = B by reducing (A|B) into Reduced Row Echelon Form (RREF) is called **Gauss-Jordan Reduction Method.**

Example 2

Solve the following system of linear equations using Gaussian Elimination Method.

$$x+y+z=1$$
$$x-y-z=5$$
$$x+y-z=3$$

Solution

The augmented matrix of the system is $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 5 \\ 1 & 1 & -1 & 3 \end{array}\right)$.

Reduce this matrix into a matrix in Row Echelon Form using appropriate elementary row operations as follows.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 5 \\ 1 & 1 & -1 & 3 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 4 \\ 0 & 0 & -2 & 2 \end{array}\right)$$

and the last matrix is in Row Echelon Form.

Thus, the systems with augmented matrices $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 5 \\ 1 & 1 & -1 & 3 \end{array}\right)$ and $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 4 \\ 0 & 0 & -2 & 2 \end{array}\right)$ are equivalent systems and hence they have the same solution set.

Then, the solution set of the system can be obtained using the augment matrix $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 4 \\ 0 & 0 & -2 & 2 \end{array}\right)$ and the system with augmented matrix $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 4 \\ 0 & 0 & -2 & 2 \end{array}\right)$ is

$$\begin{aligned} x + y + z &= 1 \\ -2y - 2z &= 4 \\ -2z &= 2 \end{aligned}$$

From the last equation, $-2z = 2$, we have $z = -1$.

Consider the second equation, $-2y - 2z = 4$, with $z = -1$ from above and solving for y gives us that $y = -1$.

Use the first equation, $x + y + z = 1$ and $y = -1$ and $z = -1$ from above to solve for x.

Then, solving for x gives you that, $x = 3$.

Therefore, the solution set of the given system is $\{(3, -1, -1)\}$.

Example 3

Solve the following system of linear equations.

$$\begin{aligned} x + 2y - z &= 0 \\ 2x + y - z &= 3 \end{aligned}$$

Solution

The augmented matrix of the system is $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & -1 & 3 \end{array}\right)$.

Then reduce the augmented matrix of the system using appropriate elementary row operations as follows.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & -1 & 3 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 1 & 3 \end{array}\right)$$ and the last matrix is in REF.

Therefore, the given system is equivalent to the system with augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 1 & 3 \end{array}\right)$$

That is, the given system is equivalent to the system

$$\begin{aligned} x + 2y - z &= 0 \\ -3y + z &= 3 \end{aligned}$$

From the second equation, solving for z in terms of y, you get $z = 3y + 3$.

From the first equation, $x = -2y + z = -2y + 3y + 3 = y + 3$.

Therefore, the system has infinitely many solutions and the solution set is given by:

$\{(y+3, y, 3y+3) : y \in \mathbb{R}\}$.

Exercise 3.19

Solve each of the following systems of linear equations using Gaussian Elimination Methods.

a. $\begin{aligned} x + y &= 5 \\ x - 2y &= -4 \end{aligned}$

b. $\begin{aligned} x + y + z &= 3 \\ x - 2y + 3z &= 1 \\ 2x + y - z &= 2 \end{aligned}$

c. $\begin{aligned} x - y - z &= 0 \\ 2x + y + 2z &= 3 \end{aligned}$

3.7 Inverse of a Square Matrix

Activity 3.24

Let $A = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}$. Then find both AB and BA.

In your responses in Activity 3.24, you have observed that $AB = I_2 = BA$. Then matrix B is called an inverse for matrix A.

Definition 3.25

Let A be a square matrix of order n. Then A is said to be invertible (or has an inverse), if there exists a square matrix B of order n such that $AB = I_n = BA$. If such matrix B exists, then it is called an inverse of A.

Let A be an invertible matrix of order n. If B and C are inverses of A, then $BA = I_n = AC$.

Using the associatively property of matrix multiplication, we have

$$B = BI_n = B(AC) = (BA)C = I_nC = C.$$

This implies, $B = C$.

Thus, if A is an invertible matrix, then its inverse is unique.

Notation

If a square matrix A is invertible, then its inverse is unique and it is denoted by A^{-1}.

Example 1

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Find the inverse of A (if it exists).

Solution

If A has an inverse, say $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we have $AA^{-1} = I_2$.

This implies $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 2c & 2d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Then, by the equality of matrices, $a = 1, b = 0, c = 0$ and $2d = 1 \Rightarrow d = \frac{1}{2}$.

Therefore, A is invertible and $A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$.

Example 2

Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$. Find its inverse if A is invertible.

Solution

If A has an inverse, say $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we have $AA^{-1} = I_2$.

This implies $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ 2a+2c & 2b+2d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Thus, we have two systems of linear equations. The first system of linear equations is

$$a + c = 1$$

$$2a + 2c = 0$$

The second system of equations is

$$b + d = 0$$

$$2b + 2d = 1$$

But each one of these two linear systems has no solution.

Therefore, matrix A has no inverse.

Exercise 3.20

Find the inverse of each of the following matrices, if it exists.

a. $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ b. $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ c. $\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$

Properties of Inverses of Matrices

Activity 3.25

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$. Then find

a. $A^{-1}, (A^{-1})^{-1}$ and B^{-1} b. $(AB)^{-1}$ and $B^{-1}A^{-1}$ c. $(A^t)^{-1}$ and $(A^{-1})^t$

In your responses in Activity 3.25, you have observed that $\left(A^{-1}\right)^{-1} = A$, $(AB)^{-1} = B^{-1}A^{-1}$ and $(A^t)^{-1} = (A^{-1})^t$.

These properties are the following properties of invertible matrices.

Properties of Invertible Matrices

Let A and B be invertible matrices of the same size. Then

a. A^{-1} is invertible and $(A^{-1})^{-1} = A$.

b. A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

c. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Example 3

Let A and B be two square matrices of order 2 such that $A^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Then, find the inverses of the matrices A^{-1}, AB and A^t.

Solution

a. Given that A is an invertible matrix, A^{-1} is invertible and $\left(A^{-1}\right)^{-1} = A$ and computing $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1}$ we get $A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$.

b. Since A and B are invertible matrices, AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

c. Because it is given that A is invertible, A^t is invertible and

$$(A^t)^{-1} = (A^{-1})^t = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^t = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Definition 3.26

An invertible matrix is also called a nonsingular matrix and a matrix that is not invertible is called a singular matrix.

Exercise 3.21

Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then, determine

a. $(A^{-1})^t$ b. $(A^t)^{-1}$ c. $(AB)^{-1}$ d. $B^{-1}A^{-1}$ e. $(A^{-1})^{-1}$

Invertibility of a Matrix and Elementary Row Operations

Activity 3.26

Consider the matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

a. Find the matrix obtained by applying the elementary row operation $R_1 \to R_1 - 2R_2$ on A.

b. Let B be a matrix obtained by applying the elementary row operation $R_1 \to R_1 - 2R_2$ on I_2.

c. Find the products AB and BA.

From your solution of (c) in Activity 3.26, observe that B is the inverse of A and B is obtained from A by applying the same elementary row operation that transforms A to I_2.

Procedures on How to Determine Invertibility of a Matrix Using Elementary Row Operations

Let A be a square matrix of size n.

Step 1: Augment A with I_n to form the $n \times 2n$ matrix $(A|I_n)$;

Step 2: Apply appropriate elementary row operations and reduce the matrix $(A|I_n)$ to a matrix in Row Echelon Form, say $(B|C)$.

Step 3: If $B = I_n$ in Step 2, then A is invertible and $A^{-1} = C$, otherwise A is not invertible.

Example 4

Using elementary row operations, determine the invertibility of each of the following matrices.

a. $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ b. $B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ c. $C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

Solution

a. First augment A with I_2 and obtain $(A|I_2) = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right)$.

Then, reduce this matrix to a matrix in Row Echelon Form using appropriate elementary row operations as follows:

$$(A|I_2) = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right).$$

Then, the last matrix is in Row Reduced Echelon Form and A is reduced into I_2.

Therefore, A is invertible with its inverse is given by:

$$A^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

b. Augment B with I_2 and obtain $(B|I_2) = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right)$.

Then, reduce it to a matrix in Row Echelon Form using appropriate elementary row operations as follows.

$$(B|I_2) = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \to -\frac{1}{2}R_2} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left(\begin{array}{cc|cc} 1 & 2 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right)$$

The last matrix is in Row Reduced Echelon Form, but B is not reduced into I_2.

Hence, B is not invertible.

c. Augment C with I_3 and obtain $(C|I_3) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$.

$$(C|I_3) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \to R_1 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 + R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

The last matrix is in Reduced Row Echelon Form and C is reduced into I_3.

Hence, C is invertible and $C^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$.

Exercise 3.22

Using appropriate elementary row operations determine the invertibility of each of the following matrices and for those that are invertible, find their inverses.

a. $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ b. $B = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$ c. $C = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

3.8 Applications

In this section, different applications of matrices will be considered.

Example 1

In a triangle, the smallest angle measures 10^o more than half of the largest angle. The middle angle measures 12^o more than the smallest angle. Find the measure of each angle of the triangle.

Solution

Let x be the measure of the smallest angle, y be the measure of the middle angle and z be the measure of the largest angle of the triangle.

Then, from the given information and the sum of the measures of the three angles of a triangle is 180°, you have the following system of linear equations.

$$x + y + z = 180$$
$$x = \frac{1}{2}z + 10$$
$$y = 12 + x.$$

The system can be written as

$$x + y + z = 180$$
$$x - \frac{1}{2}z = 10$$
$$-x + y = 12.$$

and the augmented matrix of the system is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 180 \\ 1 & 0 & -0.5 & 10 \\ -1 & 1 & 0 & 12 \end{array}\right)$$

Apply appropriate elementary row operations to reduce the augmented matrix into REF as follows.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 180 \\ 1 & 0 & -0.5 & 10 \\ -1 & 1 & 0 & 12 \end{array}\right) \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 180 \\ 0 & -1 & -1.5 & -170 \\ 0 & 2 & 1 & 192 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 180 \\ 0 & -1 & -1.5 & -170 \\ 0 & 0 & -2 & -248 \end{array}\right)$$

and this matrix is in Row Echelon Form.

Thus, the given system is equivalent to the system

$$\begin{aligned} x + y + z &= 180 \\ -y - \frac{3}{2}z &= -170 \\ -2z &= -148 \end{aligned}$$

From the last equation, you get $-2z = -148 \Rightarrow z = 74$.

From the second equation $-y - \frac{3}{2}z = -170$, we have $-y = -170 + \frac{3}{2}(74) = -59$.

This implies $y = 59$.

Then, from the last equation, $x + y + z = 180$, solving for x, gives us

$$x = 180 - (y + z) = 180 - (59 + 74) = 47.$$

Therefore, the three angles of the given triangle are 47^0, 59^0 and 74^0.

Example 2

The perimeter of a given triangle is 30 cm. The shortest side is 4 cm shorter than the longest side of the given triangle. The longest side is 6 cm less than the sum of the other two sides of the given triangle. Find the length of each side of the triangle.

Solution

Let a, b and c be the shortest, the middle and the longest sides of the given triangle respectively. Then, from the given information, you have the following system of linear equations:

$$\begin{aligned} a+b+c&=30\\ c&=a+4\\ c+6&=a+b \end{aligned}$$

Then, the given set of linear equations is equivalent to the following system of linear equations:

$$\begin{aligned} a+b+c&=30\\ a-c&=-4\\ a+b-c&=6 \end{aligned}$$

and the augmented matrix of the given system is $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 1 & 0 & -1 & -4 \\ 1 & 1 & -1 & 6 \end{array}\right)$.

Use appropriate elementary row operation to reduce the augmented matrix to a matrix in REF as follows:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 1 & 0 & -1 & -4 \\ 1 & 1 & -1 & 6 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 0 & -1 & -2 & -34 \\ 0 & 0 & -2 & -24 \end{array}\right)$$

The last matrix is in REF and the given system has the same solution as the system:

$$\begin{aligned} a+b+c&=30\\ -b-2c&=-34\\ -2c&=-24 \end{aligned}$$

From the last equation, you have $-2c=-24 \Rightarrow c=\frac{-24}{-2}=12$ and from the second equation,

$b=34-2c=34-2(12)=34-24=10.$

Then, from the first equation $a = 30 - (b + c) = 30 - 22 = 8$.

Therefore, the sides of the triangle are 8cm, 10cm and 12cm.

Example 3

If a 30% salt solution is to be mixed with a 20% salt to form a mixture of a 25% salt solution of 500 liters, how much of each is needed?

Solution

Let x represent the amount of 30% salt solution needed and y be the amount of 20% salt solution needed.

The total amount of the mixture must be 500 liters.

Thus, $x + y = 500$

The amount of salt in the end result is 25% of 500 liters, which is $0.25 \times 500 = 125$.

That is, $0.3x + 0.2y = 125$.

Then, we have the following system of linear equations:

$$x + y = 500$$
$$0.3x + 0.2y = 125$$

The augmented matrix of the system is $\left(\begin{array}{cc|c} 1 & 1 & 500 \\ 0.3 & 0.2 & 125 \end{array}\right)$.

$$\left(\begin{array}{cc|c} 1 & 1 & 500 \\ 0.3 & 0.2 & 125 \end{array}\right) \xrightarrow{R_2 \to R_2 - 0.3R_1} \left(\begin{array}{cc|c} 1 & 1 & 500 \\ 0 & -0.1 & -25 \end{array}\right)$$

This implies the given system is equivalent to the system

$$x + y = 500$$
$$-0.1y = -25$$

Thus, $-0.1y = -25 \Rightarrow y = \dfrac{-25}{-0.1} = 250$ and $x = 500 - y = 500 - 250 = 250$.

Exercise 3.23

1. In a triangle, the smallest angle measures 10^o less than half of the largest angles. The middle angle measures 20^o more than the smallest angle. Find the measure of each angle of the triangle.
2. The perimeter of a triangle is 55 cm. The measure of the shortest side is 8 cm less than the middle side. The measure of the longest side is 1 cm less than the sum of the other two sides. Find the lengths of the sides.
3. A chemist has 6% salt solution and 12% salt solution. How much of the 6% salt solution and 12% salt solution must be added to get a 300 g of 10% salt solution?

Problem Solving

1. The perimeter of a triangle is 50 m. The longest side of the triangle measures 20 m. more than the shortest side. The middle side is 3 times the measure of the shortest side. Find the lengths of the three sides of the triangle.
2. A train travels 700 kilometers in the same time that a truck travels 500 kilometers. Find the speed of each vehicle if the train's average speed is 10 kilometers per hour faster than the truck's speed.
3. Almaz, Birkti, and Chaltu work in a boutique. One day the three had combined sales of Birr 14800. Almaz sold Birr 1200 more than Birkti. Birkti and Chaltu combined sold Birr 2800 more than Almaz. How much did each person sell on that particular day?

Summary

1. A rectangular array of numbers arranged in rows and columns is called a matrix.
2. A matrix with only one column is called a column matrix. .
3. A matrix with only one row is called a row matrix.
4. A matrix with equal number of rows and columns is called a square matrix.
5. A matrix with all entries 0 is called a zero matrix.
6. A square matrix that has zeros everywhere except possibly along the main diagonal is called diagonal matrix.
7. A diagonal matrix where all elements of the diagonal are equal is called a scalar matrix.
8. The diagonal matrix where all the elements of the diagonal are 1 is called the identity matrix.
9. A square matrix whose elements below the main diagonal are all zero is called an upper triangular matrix.
10. A square matrix whose elements above the main diagonal are all zero is called a lower triangular matrix.
11. Let A $= (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ be two matrices and k be a real number. Then
    a. The sum of A and B is given by, $\mathrm{A}+\mathrm{B}=(a_{ij}+b_{ij})_{m\times n}$.
    b. The difference A and B is given by $\mathrm{A}-\mathrm{B}=\left(a_{ij}-b_{ij}\right)_{m\times n}$
    c. The scalar multiple of A by k is defined by $k\mathrm{A}=\left(ka_{ij}\right)_{m\times n}$

    12Let $A = \left(a_{ij}\right)_{m\times p}$ and $\mathrm{B}=\left(b_{kl}\right)_{p\times n}$ be matrices, where m, n and p are positive integers. The product of A and B, denoted by AB, is the $m \times n$ matrix AB $= \left(c_{ij}\right)_{m\times n}$. where, $c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots\ldots+a_{ip}b_{pj}$, for $i = 1, \ldots\ldots m$ and $j = 1, \ldots n$.

12. Given a matrix A, the matrix obtained by interchanging the rows of A to columns and the columns of *A* to rows is called the transpose of A and it is denoted by A^t.
13. The following operations are called Elementary Row operations.
    I. **Swapping:** Interchanging two rows of a matrix.
    II. **Scaling:** Multiplying a row of a matrix by a non-zero constant.
    III. **Pivoting:** Adding a constant multiple of one row of a matrix on another row.
14. A given matrix is said to be in Row Echelon Form, if the following are satisfied:
    a. if there are any rows with no leading entries (rows having zeros entirely) they are at the bottom.
    b. the leading entry (the first non-zero entry) in each row after the first is to the right of the leading entry in the previous row.
15. A is said to be in **Reduced Row Echelon Form (RREF)** if A is in **Row Echelon Form (REF)** and the leading element in each nonzero row is 1 and it is the only nonzero number in its column.

Review Exercise

1. Find the size of each of the following matrices.

a. $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & -3 \end{pmatrix}$ b. $B = \begin{pmatrix} -1 & 2 & -3 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ c. $C = \begin{pmatrix} -1 & 2 \\ 0 & 3 \\ 3 & -1 \end{pmatrix}$

d. $D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 3 \\ 3 & -2 & 3 \\ 2 & 1 & 0 \end{pmatrix}$ e. $E = \begin{pmatrix} 0 & 1 & -3 & -1 \end{pmatrix}$ f. $F = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$

2. Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \end{pmatrix}$. Then find each of the following matrices.

a. $A - 3C$ b. $AB - CD$ c. $2B + 3D$ d. $B^tC + D^tA$

3. Identify each of the following matrices as symmetric, skew-symmetric, diagonal, scalar, upper triangular or lower triangular matrices.

a. $A = \begin{pmatrix} -1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ b. $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ c. $C = \begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 3 \\ 1 & -3 & 0 \end{pmatrix}$

d. $D = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 1 & 4 \end{pmatrix}$ e. $E = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ 4 & 2 & -5 \end{pmatrix}$ f. $F = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

4. Find the reduced row echelon form of each of the following matrices.

a. $A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 1 & -1 \end{pmatrix}$ b. $B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \end{pmatrix}$

5. Solve each of the following systems of linear equations using Gaussian Elimination Method.

a. $\begin{aligned} x - y &= 1 \\ 2x + y &= 2 \end{aligned}$ b. $\begin{aligned} x + y + z &= 1 \\ y + z &= 3 \\ x - 2y + z &= -5 \end{aligned}$

6. Determine the inverse of each of the following matrices, if it exists.

a. $A = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$ b. $B = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$

c. $C = \begin{pmatrix} 1 & 2 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$ d. $D = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -3 & 3 \\ -1 & 2 & -2 \end{pmatrix}$

7. A farmer has two types of milk, one that is 20% butterfat and another which is 15% butterfat. How much of each should he use to end up with 50 gallons of 20% butterfat?

UNIT 4

DETERMINANTS AND THEIR PROPERTIES

Unit Outcomes

By the end of this unit, you will be able to:

- Know minor and cofactor of a matrix.
- Evaluate determinant of a matrix.
- Understand properties of a determinant.
- Apply principles of determinants to compute inverse of a matrix.
- Use Cramer's rule to solve system of linear equations.
- Apply determinant concepts to solve real-world situations.

Unit Contents

KEY Words

- **Adjoint**
- **cofactor**
- **cramer's rule**
- **determinant**
- **inverse**
- **minor**

Introduction

Every square matrix can be associated with a number called its determinant. Determinants occur in many mathematical topics. The coefficients in system of linear equations are often represented by matrices and the determinant can be used to solve systems of linear equations, if the coefficient matrix is an invertible matrix.

In this unit, determinant of a matrix; basic properties of determinant will be discussed and some applications of determinants in solving systems of linear equations will also be given.

4.1 Determinants of Matrices of Order 2

Activity 4.1

Consider the square matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of order 2. Suppose $ad - bc \neq 0$. Then

a. show that A is invertible and find A^{-1}.

b. solve the following system of linear equations:

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

From Activity 4.1, you have found that

a. A is invertible and its inverse is given by $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ and

b. The solution set of the given system is $\left\{\left(\frac{ed-fb}{ad-bc}, \frac{af-ce}{ad-bc}\right)\right\}$.

In both cases, the number $ad - bc$ is involved and $ad - bc$ is called the determinant of A.

The determinant of a square matrix of order 2 is defined as follows.

Definition 4.1

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a square matrix of order 2. The determinant of A, denoted by det(A) or |A|, is the number defined by $\det(A) = a_{11}a_{22} - a_{12}a_{21}$.

Example 1

Find the determinant of $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Solution

Using the definition, $\det(A) = 1(4) - 2(3) = 4 - 6 = -2$

Example 2

Let $A=\begin{pmatrix}1 & x\\ 2 & x+1\end{pmatrix}$. If $\det(A)=0$, then find x.

Solution

If $\det(A)=0$, then $1(x+1)-2x=0 \Rightarrow 1-x=0$.

This implies, $x=1$.

Example 3

Find the determinant of $I_2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

Solution

The determinant of the identity matrix for multiplications is given as:

$$\det(I_2)=\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}=1(1)-0(0)=1-0=1.$$

Exercise 4.1

1. Determine the determinant of each of the following matrices.

   a. $A=\begin{pmatrix}1 & 2\\ 1 & 4\end{pmatrix}$ b. $B=\begin{pmatrix}5 & 2\\ 3 & 1\end{pmatrix}$ c. $C=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$

2. Determine the determinant of each of the following matrices.

   a. $\begin{pmatrix}1 & 2\\ x & 2x\end{pmatrix}$ b. $\begin{pmatrix}x & 0\\ 0 & x\end{pmatrix}$ c. $\begin{pmatrix}1 & a\\ 0 & 2\end{pmatrix}$

3. Let $A=\begin{pmatrix}3 & x-1\\ 1 & x\end{pmatrix}$. If $\det(A)=0$ then determine the value(s) of x.

4.2 Minors and Cofactors of Elements of Matrices

Minors of Elements of Matrices

Activity 4.2

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Then find

a. matrix B obtained by crossing-out the 2nd row and the 1st column from A.

b. det(B)

c. matrix C obtained by crossing-out the 1st row and the 2nd column from A.

d. det(C)

From your responses for Activity 4.2, observe that

(i) both B and C are 2×2 matrices;

(ii) the determinant of B is called the minor of a_{21}, the element in the intersection of the second row and the first column of A;

(iii) the determinant of C is called the minor of a_{12}, the element in the intersection of the first row and the second column of A.

Note

If $A = (a)$ is a 1×1 matrix, then the determinant of A is defined by $\det(A) = a$.

Definition 4.2

Let n be a positive integer greater than or equal to 2 and $A = (a_{ij})_{n\times n}$ be a square matrix of order n.

Suppose we know how to define the determinant of any $(n-1)\times(n-1)$ matrix.

Let (i, j), for $i = 1, \ldots, n$ and $j = 1, \ldots, n$, be an ordered pair of positive integers.

Cross out the i^{th} row and j^{th} column of A and obtain an $(n-1)\times(n-1)$ matrix and this matrix is denoted by A_{ij}.

That is,

$$A_{ij} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}_{(n-1)\times(n-1)}$$

Then $\det(A_{ij})$ is called the minor of the entry a_{ij} of A, denoted by M_{ij}.

Example 1

Determine the minors of all the entries of matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 3 & -1 & 5 \end{pmatrix}$.

Solution

i. $A_{11} = \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}$ $\quad A_{12} = \begin{pmatrix} 0 & 0 \\ 3 & 5 \end{pmatrix}$ $\quad A_{13} = \begin{pmatrix} 0 & 1 \\ 3 & -1 \end{pmatrix}$

$A_{21} = \begin{pmatrix} 2 & 1 \\ -1 & 5 \end{pmatrix}$ $\quad A_{22} = \begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix}$ $\quad A_{23} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$

$A_{31} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad A_{32} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad A_{33} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

ii. The minors of A are

$$|A_{11}| = \begin{vmatrix} 1 & 0 \\ -1 & 5 \end{vmatrix} = 5 \qquad |A_{12}| = \begin{vmatrix} 0 & 0 \\ 3 & 5 \end{vmatrix} = 0 \qquad |A_{13}| = \begin{vmatrix} 0 & 1 \\ 3 & -1 \end{vmatrix} = -3,$$

$$|A_{21}| = \begin{vmatrix} 2 & 1 \\ -1 & 5 \end{vmatrix} = 11 \qquad |A_{22}| = \begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} = 2 \qquad |A_{23}| = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -7,$$

$$|A_{31}| = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -1 \qquad |A_{32}| = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0 \qquad |A_{33}| = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1.$$

Cofactors of Elements of Matrices

Activity 4.3

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Then compute $(-1)^{i+j}\det(A_{ij})$ for every (i, j) pair of positive integers where $i = 1, 2, 3$ and $j = 1, 2, 3$.

From your responses in Activity 4.3, the values $(-1)^{i+j}\det(A_{ij})$ is called the cofactor of the entry a_{ij} for every pair (i, j) of positive integers for $i = 1,2,3$ and $j = 1,2,3$.

Definition 4.3

Let n be a positive integer greater than or equal to 2 and $A = (a_{ij})_{n\times n}$ be a square matrix of order n. Then $C_{ij} = (-1)^{i+j}\det(A_{ij})$ is called the cofactor of a_{ij}, for every pair of positive integers (i, j) for $1 \le i \le n$ and $1 \le j \le n$.

Example 2

Determine the cofactors of all the entries of the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 3 & -1 & 5 \end{pmatrix}$.

Solution

From Example 1, the minors of all the entries of A are:

$M_{11} = |A_{11}| = 5$ $\quad M_{12} = |A_{12}| = 0$ $\quad M_{13} = |A_{13}| = -3$

$M_{21} = |A_{21}| = 11$ $\quad M_{22} = |A_{22}| = 2$ $\quad M_{23} = |A_{23}| = -7$

$M_{31} = |A_{31}| = -1$ $\quad M_{32} = |A_{32}| = 0$ $\quad M_{33} = |A_{33}| = 1.$

Then the cofactors of all the entries of A are:

$C_{11} = (-1)^{1+1}|A_{11}| = 5$ $\quad C_{12} = (-1)^{1+2}|A_{12}| = 0$ $\quad C_{13} = (-1)^{1+3}|A_{13}| = -3$

$C_{21} = (-1)^{2+1}|A_{21}| = -11$ $\quad C_{22} = (-1)^{2+2}|A_{22}| = 2$ $\quad C_{23} = (-1)^{2+3}|A_{23}| = 7$

$C_{31} = (-1)^{3+1}|A_{31}| = -1$ $\quad C_{32} = (-1)^{3+2}|A_{32}| = 0$ $\quad C_{33} = (-1)^{3+3}|A_{33}| = 1.$

Exercise 4.2

Find the minors and the cofactors of all the entries of the following matrices.

a. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ $\qquad$ b. $B = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{pmatrix}$

4.3 Determinants of Matrices of Order 3

Activity 4.4

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 4 & 4 \end{pmatrix}$. Then, determine

a. $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ $\qquad$ b. $a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$.

From your responses in Activity 4.3, observe that the answers for both questions are the same number and this number is called the determinant of A.

Definition 4.4

Let $A = (a_{ij})_{3\times3}$ be a square matrix of order 3. Then the determinant of A, det(A) or |A|, is the number defined by

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

and this sum is called the cofactor expansion of the determinant along the first row.

It is possible to express determinant of a matrix A as a cofactor expansion along a column of A and gives you the same result, but in this subunit, cofactor expansion of the determinant along a row only is considered.

Example 1

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Then find the determinant of A using a cofactor expansion along the first row.

Solution

First find the cofactors C_{11}, C_{12} and C_{13} as:

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = 45 - 48 = -3,\ C_{12} = (-1)^{1+2}\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = -(36-42) = 6$$

and $C_{13} = (-1)^{1+3}\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 32 - 35 = -3.$

Then the determinant of A is given by:

$$|A| = 1C_{11} + 2C_{12} + 3C_{13} = 1(-3) + 2(6) + 3(-3) = -3 + 12 - 9 = 0.$$

That is, $|A| = 0$.

Example 2

Find the determinant of $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Solution

Using a cofactor expansion along the first row, the determinant of I_3 is given by:

$$\det(I_3) = 1(-1)^{(1+1)}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 0(-1)^{(1+2)}\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + 0(-1)^{(1+3)}\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 1 - 0 + 0 = 1$$

Therefore, the determinant of I_3 is 1.

Exercise 4.3

1. Let $A = \begin{pmatrix} 2 & 7 & 9 \\ 3 & 5 & 6 \\ 7 & 4 & 1 \end{pmatrix}$. Then compute the determinant of A using a cofactor expansion along the first row.
2. Compute the determinant of the matrix $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Cofactor Expansion of Determinant along Any Row

Activity 4.5

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 3 & 3 & 2 \end{pmatrix}$. Then evaluate each of the following.

a. $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

b. $a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$

c. $a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$

In your responses in Activity 4.5, you have observed that, the values in (a), (b) and (c) are the same and this number is the determinant of A.

The results in Activity 4.5 tell us that the determinant does not depend on the choice of the row as stated in Theorem 4.1.

Theorem 4.1

Let $A = (a_{ij})_{3\times 3}$ be a square matrix of order 3. Then, the determinant of A, det(A), can be expressed as a cofactor expansion along any row of A.

That is, $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3}$ for each $i = 1, 2, 3$.

Example 3

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Then, find the determinant of A using cofactor expansion along the three rows of A.

Solution

In Example 1 of these section, you have seen that using cofactor expansion of the determinant along first row, $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 0$.

a. The determinant of A using the cofactor expansion along the second row is given by:

$$\det(A) = (-1)^{(2+1)} 4\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + (-1)^{(2+2)} 5\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + (-1)^{(2+3)} 6\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$$

$$= -4(18-24) + 5(9-21) - 6(8-14) = 24 - 60 + 36 = 0.$$

b. The determinant of A using a cofactor expansion along the third row is given by:

$$\det(A) = (-1)^{(3+1)}7\begin{vmatrix}2 & 3\\5 & 6\end{vmatrix} + (-1)^{(3+2)}8\begin{vmatrix}1 & 3\\4 & 6\end{vmatrix} + (-1)^{(3+3)}9\begin{vmatrix}1 & 2\\4 & 5\end{vmatrix}$$

$$= 7(12-15) - 8(6-12) + 9(5-8)$$

$$= 7(-3) - 8(-6) + 9(-3) = -21 + 48 - 27 = 0$$

Thus, the determinant of A using the cofactor expansion along any of its three rows is the same: that is, |A| = 0.

Note

Let $A = (a_{ij})_{n\times n}$ be a square matrix of order n.

a. The expression $(-1)^{i+j}$ in the expansion of the determinant determines the algebraic sign in the position (i, j) of a square matrix of order n and these signs form a checkerboard pattern of " + " and " − " that has + in the (1, 1) position. The patterns for $n = 2$ and 3 are given as follows.

$$\underset{n=2}{\begin{pmatrix}+ & -\\ - & +\end{pmatrix}} \text{ and } \underset{n=3}{\begin{pmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{pmatrix}}$$

b. In determining the determinant of a square matrix by cofactor expansion along a row, the best strategy is to use a cofactor expansion along a row with large number of zeros.

Example 4

Compute the determinant of the matrix $A = \begin{pmatrix}1 & 2 & 1\\12 & 0 & 0\\-5 & 11 & 0\end{pmatrix}$.

Solution

As the second row has two zero entries, more zero entries than the other two rows, you can use a cofactor expansion along the second row.

$$\begin{vmatrix} 1 & 2 & 1 \\ 12 & 0 & 0 \\ -5 & 11 & 0 \end{vmatrix} = -12\begin{vmatrix} 2 & 1 \\ 11 & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ -5 & 0 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ -5 & 11 \end{vmatrix}$$

$$= -12\begin{vmatrix} 2 & 1 \\ 11 & 0 \end{vmatrix} = (-12)\times(0-11) = 132$$

Therefore, the determinant of A is 132.

Exercise 4.4

1. Let $A = \begin{pmatrix} 1 & 7 & 2 \\ 0 & 5 & 2 \\ -1 & 4 & 1 \end{pmatrix}$. Then compute the determinant of A using a cofactor expansion along all the three rows.
2. Compute the determinant of each of the following matrices by using cofactor expansion.

   a. $B = \begin{pmatrix} 2 & 1 & 0 \\ 3 & -2 & 2 \\ 1 & 0 & 2 \end{pmatrix}$ b. $C = \begin{pmatrix} 0 & 2 & 1 \\ 4 & 1 & 1 \\ 3 & 2 & 0 \end{pmatrix}$

3. Find the determinant of each of the following matrices.

   a. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ b. $\begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -4 \end{pmatrix}$ c. $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 5 & 6 & 3 \end{pmatrix}$

4.4 Properties of Determinants

Determinants of Triangular Matrices

Activity 4.6

Find the determinant of each of the following matrices.

a. $A = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$ b. $B = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$

From your responses in Activity 4.6, observe that matrices A and B are triangular matrices and

a. the determinant of matrix A is the product of its diagonal entries;

b. the determinant of matrix B is the product of its diagonal entries.

These properties are true in general and are stated as follows.

Theorem 4.2

Let n be a positive integer greater than or equal to 2 and $A = (a_{ij})_{n\times n}$. If A is a triangular matrix, then the determinant of A is the product of its diagonal elements.

That is, if $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$ or $A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n-1)1} & a_{(n-2)2} & \dots & 0 \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$,

then $|A| = a_{11}a_{22} \dots a_{nn}$.

Note

1. If A is a diagonal matrix, then A is a triangular matrix and its determinant is the product of its diagonal entries.

   That is, if $A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 & 0 \\ 0 & a_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix}$, then $|A| = a_{11}a_{22} \dots a_{nn}$

2. For a positive integer n the matrix $I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$ is the identity matrix for matrix multiplication and it is a diagonal matrix.

   Thus, $|I_n| = 1 \times 1 \times \dots \times 1 = 1$.

Example 1

Compute the determinant of $A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$.

Solution

Using the cofactor expansion along the first row:

$$\det(A) = 2\begin{vmatrix} -1 & 4 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 4 \\ 0 & 1 \end{vmatrix} + 3\begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix}$$

$$= 2\times(-1) + (-1)\times 0 + 3\times 0 = -2$$

On the other hand, A is an upper triangular matrix.

Thus, $\det(A) = 2\times(-1)\times 1 = -2$, the product of the diagonal entries of A.

Example 2

Find the determinant of each of the following matrices:

a. $A = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{pmatrix}$ b. $B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ c. $C = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

Solution

a. A is an upper triangular matrix and $|A| = 2\times3\times1 = 6$, which is the product of the diagonal entries of A.

b. B is a lower triangular matrix and $|B| = -1\times2\times1 = -2$, which is the product of the diagonal entries of B.

c. C is a diagonal matrix and hence $|C| = 4\times(-1)\times(-2) = 8$, which is the product of the diagonal entries of C.

Exercise 4.5

Determine the determinant of each of the following matrices:

a. $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & -1 \end{pmatrix}$ b. $B = \begin{pmatrix} 2 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & -2 \end{pmatrix}$ c. $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Determinant and Elementary Row Operations

Let us consider the effects of the three elementary operations of matrices on determinants of matrices. Let us start our discussion by considering the following activity.

Activity 4.7

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then find the determinant of each of the following matrices in terms of the determinant of A.

a. $\begin{pmatrix} c & d \\ a & b \end{pmatrix}$ b. $\begin{pmatrix} 3a & 3b \\ c & d \end{pmatrix}$ c. $\begin{pmatrix} a & b \\ c-2a & d-2b \end{pmatrix}$

From your responses in Activity 4.6, observe that:

a. $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = -|A|$ (Interchanging two rows of a matrix changes the determinant by sign).

b. $\begin{vmatrix} 3a & 3b \\ c & d \end{vmatrix} = 3|A|$ (The determinant of a matrix obtained from A by multiplying a row of A by a scalar is the scalar times the determinant of A).

c. $\begin{vmatrix} a & b \\ c-2a & d-2b \end{vmatrix} = |A|$ (Adding a scalar multiple of a row of A onto another row does not change the determinant).

These three properties are the effects of elementary row operations on the determinant of A and these effects are true in general for any square matrix as stated in Theorem 4.3.

Theorem 4.3

Let A be a square matrix of order n.

a. Interchanging two rows of the matrix changes the sign of the determinant;

That is, if $A \xrightarrow{R_i \leftrightarrow R_j} B$ for $i \neq j$, then $|B| = -|A|$.

b. The determinant of a matrix obtained by multiplying one row of A by a constant gives you the same constant times the determinant of the given matrix.

That is, if, $A \xrightarrow{R_i \to kR_i} B$ and k is any constant, then $|B| = k|A|$.

c. Adding a scalar multiple of one row of A to another row of A does not change the determinant. That is, if $A \xrightarrow{R_i \to R_i + kR_j} B$ for $i \neq j$ and k is any constant, then $|B| = |A|$.

Note

One consequence that follows from property (b) of Theorem 4.3 is if you multiply a square matrix of order n by a scalar k, then the determinant of the new matrix is k^n times the determinant of the given matrix.

That is, if A is a square matrix of order n and k is a scalar, then $|kA| = k^n|A|$.

Example 3

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $\det(A) = -2$, then find the determinant of each the following matrices:

a. $B=\begin{pmatrix} a & b \\ 2a+c & 2b+d \end{pmatrix}$

b. $C=\begin{pmatrix} 2a & 2b \\ c & d \end{pmatrix}$

c. $D=\begin{pmatrix} 3a & 3b \\ 3c & 3d \end{pmatrix}$

d. $E=\begin{pmatrix} c & d \\ a & d \end{pmatrix}$

Solution

If $\det(A) = -2$, then $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = -2$.

a. $|B| = \begin{vmatrix} a & b \\ 2a+c & 2b+d \end{vmatrix} = a(2b+d) - b(2a+c) = ad - bc = -2$. Observe that, matrix B is obtained by replacing the third row of A by 2 times the first row of A plus the second row of A and $|A| = |B| = -2$.

b. $|C| = \begin{vmatrix} 2a & 2b \\ c & d \end{vmatrix} = 2a(d) - 2b(c) = 2(ad - bc) = 2\times(-2) = -4$. Observe that, matrix C is obtained by scaling the first row of A by -2 and $|C| = 2|A| = 2\times(-2) = -4$.

c. $|D| = \begin{vmatrix} 3a & 3b \\ 3c & 3d \end{vmatrix} = 3a(3d) - 3b(3c) = 3\times3(ad - bc) = 3^2\times(-2) = 9\times(-2) = -18$.

Observe that $D = 3A$ and $|D| = 3^2|A| = 9\times(-2) = -18$.

d. $|E| = \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - da = -(ad - bc) = 2$. Observe that, matrix E is obtained by swapping the first and the second rows of A and $\det(E) = -\det(A) = -(-2) = 2$.

Example 4

Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$. If det(A) = 2, then find the determinant of each of the following matrices.

a. $B = \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & k \end{pmatrix}$

b. $C = \begin{pmatrix} a & b & c \\ d & e & f \\ g+3a & h+3b & k+3c \end{pmatrix}$

c. $D = \begin{pmatrix} a & b & c \\ 5d & 5e & 5f \\ g & h & k \end{pmatrix}$

d. $E = \begin{pmatrix} 4a & 4b & 4c \\ 4d & 4e & 4f \\ 4g & 4h & 4k \end{pmatrix}$

Solution

a. Matrix B is obtained by swapping the first and the second rows of A.

Thus, $\det(B) = -\det(A) = -2$.

b. Matrix C is obtained by replacing the third row of A by 3 times the first row of A plus the third row of A. Thus, $\det(C) = \det(A) = 2$.

c. Matrix D is obtained by scaling the second row of A by 5.

Thus, $\det(D) = 5\det(A) = 5 \times 2 = 10$.

d. A is a 3×3 and $E = 4A$.

Thus, $\det(E) = \det(4A) = 4^3 \det(A) = 64 \times 2 = 128$.

Note

Using property (b) in Theorem 4.3 we can introduce many zeros in a row of a square matrix and compute the determinant of the matrix by expanding the determinant along the row with many zeros.

Example 5

Determine the determinant of the following matrix after reducing it to a matrix in REF.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix}$$

Solution

First reduce A into an upper triangular matrix using appropriate elementary row operations as follows.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{pmatrix} = B$$

The last matrix, B, is a triangular matrix and hence the determinant of B is the product of its diagonal entries.

That is, $|B| = 1 \times 1 \times (-2) = -2$.

Observe that matrix B is obtained from A by applying three successive elementary row operations. Each of these elementary row operations are of the same type; adding a scalar multiple of one row onto another row.

Thus, all these elementary row operations do not make any change on the determinant of the matrix.

Therefore, the determinants of both A and B are the same and hence

$|A| = |B| = -2$.

Exercise 4.6

1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $\det(A) = 2$, then find the determinant of each the following matrices.

   a. $\begin{pmatrix} a & b \\ a+c & b+d \end{pmatrix}$ b. $\begin{pmatrix} a & b \\ 2c & 2d \end{pmatrix}$ c. $\begin{pmatrix} -3a & -3b \\ -3c & -3d \end{pmatrix}$

2. Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$ and $\det(A) = 3$. Then, compute the determinant of each of the following matrices.

   a. $\begin{pmatrix} a & b & c \\ g & h & k \\ d & e & f \end{pmatrix}$ b. $\begin{pmatrix} 2a & 2b & 2c \\ d & e & f \\ g & h & k \end{pmatrix}$ c. $\begin{pmatrix} a & b & c \\ d & e & f \\ g+3a & h+3b & k+3c \end{pmatrix}$

3. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$. Determine the determinant of matrix B if:

   a. $A \xrightarrow{R_1 \leftrightarrow R_3} B$ b. $A \xrightarrow{R_2 \to R_2 - 2R_3} B$ c. $A \xrightarrow{R_3 \to \frac{1}{2} R_3} B$

Determinant of Product of Matrices and Determinant of a Transpose

Activity 4.8

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Then, compute each of the following.

a. $\det(A)$ and $\det(B)$. b. $\det(AB)$, $\det(A^t)$ and $\det(B^t)$

In your responses in Activity 4.8, observe that;

a. $\det(A) = \det(A^t)$, that is, the determinant of A and its transpose are equal;

b. $\det(AB) = (\det A)(\det B)$; that is, the determinant of the product of A and B, $\det(AB)$, is equal to the product of the determinant of A and the determinant of B.

In general, we have the following results about the determinant of a transpose and the determinant of product of matrices.

Theorem 4.4

Let A and B be two square matrices of order n. Then

a. $\det(A) = \det(A^t)$ b. $\det(AB) = \det(A)\det(B)$

Note

Using Property (b) of Theorem 4.4 and the associative property of matrix multiplication, if A, B and C area square matrix of order n, , then $|ABC| = |A||B||C|$.

Example 6

Let A and B be square matrices of order 3 with $|A| = 4$ and $|B| = 5$. Then compute each of the following:

a. $\det(AB)$ b. $\det(3A)$ c. $\det(2AB^t)$ d. $\det(A^5)$

Solution

a. $\det(AB) = \det(A)\det(B) = 4 \times 5 = 20$.

b. $\det(3A) = 3^3 \det(A) = 27 \times 4 = 108$.

c. $\det(2AB^t) = 2^3 \det(AB^t)$, since AB^t is a square matrix of order 3.

The matrix AB^t is the product of A and B^t.

This implies $\det(2AB^t) = 2^3 \det(A)\det(B^t) = 2^3 \det(A)\det(B)$ and

$\det(B) = \det(B^t)$.

Therefore, $\det(2AB^t) = 2^3 \det(A)\det(B) = 8 \times 4 \times 5 = 160$.

d. $\det(A^5) = (\det A)^5 = 4^5 = 1024$.

Exercise 4.7

Let A and B be square matrices of order 3 with $|A| = 2$ and $|B| = 3$. Then compute each of the following.

a. $\det(AB)$ b. $\det(2A)$ c. $\det(3AB^t)$ d. $\det(A^4)$

Determinant of Inverse of an Invertible Matrix

Let A be a square matrix of order n. If A is invertible, then $AA^{-1} = I_n$ and

$$\det(I_n) = \det(AA^{-1}) = [\det(A)][\det(A^{-1})] = 1$$

Therefore, $\det(A) \neq 0$, because the product of det A and $\det(A^{-1})$ is different from zero and hence, $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Thus, the following theorem is proved.

Theorem 4.5

If A is an invertible matrix of order n, then $\det(A)$ is not zero and the determinant of A^{-1} is given by $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Example 7

Let A and B be two invertible matrices of order 3. Suppose $\det(A) = 4$ and $\det(B) = 3$. Then find

a. $\det(A^{-1})$ b. $\det(A^{-1}B)$

Solution

a. Since A is invertible and $\det(A) = 4 \neq 0$, then $\det(A^{-1}) = \dfrac{1}{\det(A)} = \dfrac{1}{4}$;

b. $\det(A^{-1}B) = \det(A^{-1})\det(B) = \left(\dfrac{1}{\det(A)}\right)\det(B) = \dfrac{1}{4} \times 3 = \dfrac{3}{4}$

Exercise 4.8

Let A and B be 3 × 3 matrices with det(A) = 2 and det(B) = 5. Then find

a. $\det(A^{-1})$

b. $\det\left(A^{-1}B\right)$

c. $\det(B^{-1})$

d. $\det(AB^{-1})$

4.5 Inverse of a Square Matrix of Order 2 and 3

There are different applications of determinants. One of these applications that is considered in this section is invertibility of a matrix.

Adjoint of a square Matrix

Activity 4.9

Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 4 & 1 & 4 \end{pmatrix}$. Then find

a. the cofactors of all the entries of A.

b. matrix B defined by $B = \left(C_{ij}\right)^{t}_{3\times 3}$.

The matrix B you have obtained in Activity 4.9 is called the adjoint of A and if A is invertible, then matrix B has a role in determining the inverse of A.

Definition 4.5

Let $A = (a_{ij})_{n\times n}$ be a square matrix of order n and $C_{ij} = (-1)^{(i+j)}\det(A_{ij})$ be the cofactor of the element a_{ij} for $i = 1, \ldots, n$ and $j = 1, \ldots, n$. Then, the matrix

$$(C_{ij})^{t}_{n\times n} = \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}$$

is called adjoint of A and it is denoted by Adj(A).

Example 1

Determine the adjoint of $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Solution

$A_{11} = (4)$ $\quad A_{12} = (3)$ $\quad A_{21} = (2)$ $\quad A_{22} = (1)$.

Thus, the cofactors of all the elements A are:

$$C_{11} = (-1)^{1+1} \det(A_{11}) = 4 \qquad C_{12} = (-1)^{1+2} \det(A_{12}) = -3$$

$$C_{21} = (-1)^{2+1} \det(A_{21}) = -2 \qquad C_{22} = (-1)^{2+2} \det(A_{22}) = 1.$$

Therefore, the adjoint of A is

$$\text{Adj}(A) = \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}^t = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

Example 2

Determine the adjoint of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 4 & 3 \\ 1 & 2 & 0 \end{pmatrix}.$$

Solution

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 4 & 3 \\ 2 & 0 \end{vmatrix} = -6 \qquad C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} = 3 \qquad C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} = -2$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} = 2 \qquad C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \qquad C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = 2 \qquad C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2 \qquad C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 2$$

are the cofactors of all the entries of A.

Therefore, the adjoint of A is:

$$\text{Adj}(A)=\begin{pmatrix} C_{11} & C_{12} & C_{31} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^t = \begin{pmatrix} -6 & 3 & -2 \\ 2 & -1 & 0 \\ 2 & -2 & 2 \end{pmatrix}^t = \begin{pmatrix} -6 & 2 & 2 \\ 3 & -1 & -2 \\ -2 & 0 & 2 \end{pmatrix}$$

Exercise 4.9

1. Determine the adjoint of $A=\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$.
2. Determine the adjoint of $B=\begin{pmatrix} 1 & 2 & 3 \\ 6 & 1 & 9 \\ 4 & 0 & 5 \end{pmatrix}$.

Adjoint and Determinant

Activity 4.10

Let $A=\begin{pmatrix} 1 & 1 & 2 \\ 3 & 4 & 3 \\ 2 & 2 & 1 \end{pmatrix}$. Then find

a. $\det(A)$ b. Adj(A) c. A(Adj(A)) d. (Adj(A))A

From your responses in Activity 4.10, observe that

$(\text{Adj}(A))A=\det(A)I_3=A(\text{Adj}(A))$.

In general, given a square matrix of order n, $A(\text{Adj}(A)) = \det(A)\, I_n = (\text{Adj}(A))A$.

If det(A) is not equal to zero, then

$$A\left(\frac{1}{\det(A)}\text{Adj}(A)\right)=I_n=\left(\frac{1}{\det(A)}\text{Adj}(A)\right)A$$

This implies, A is invertible and its inverse is given by:

$$A^{-1}=\frac{1}{\det(A)}(\text{Adj}(A))$$

Thus, the following result is proved.

Theorem 4.6

Let A be a square matrix of order n. Then, $A\left(\mathrm{Adj}(A)\right)=\left(\mathrm{Adj}(A)\right)A=\det(A)I_n$ and if the determinant of A is not zero, then A is invertible and its inverse is given by

$$A^{-1}=\frac{1}{|A|}(\mathrm{Adj}(A)).$$

Example 3

Let $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$. Find the inverse of A (if it exists).

Solution

a. From Example 1, $\mathrm{Adj}(A)=\begin{pmatrix}4 & -3\\ -2 & 1\end{pmatrix}^t=\begin{pmatrix}4 & -2\\ -3 & 1\end{pmatrix}$.

b. $\det(A)=4-6=-2\neq 0$.

This implies, A is invertible and its inverse is given by

$$A^{-1}=\frac{1}{\det(A)}\mathrm{Adj}(A)=-\frac{1}{2}\begin{pmatrix}4 & -2\\ -3 & 1\end{pmatrix}=\begin{pmatrix}-2 & 1\\ \frac{3}{2} & -\frac{1}{2}\end{pmatrix}.$$

Example 4

Let $A=\begin{pmatrix}1 & 2 & 1\\ 1 & 4 & 3\\ 1 & 2 & 0\end{pmatrix}$. Find the inverse of A (if it exists).

Solution

From Example 2, $\text{Adj}(A)=\begin{pmatrix}-6 & 2 & 2\\ 3 & -1 & -2\\ -2 & 0 & 2\end{pmatrix}$ and

$$|A|=\begin{vmatrix}4 & 3\\ 2 & 0\end{vmatrix}-2\begin{vmatrix}1 & 3\\ 1 & 0\end{vmatrix}+\begin{vmatrix}1 & 4\\ 1 & 2\end{vmatrix}=-6+6-2=-2\neq 0.$$

Therefore, A is invertible and its inverse is given by

$$A^{-1}=\frac{1}{\det(A)}\text{Adj}(A)=-\frac{1}{2}\begin{pmatrix}-6 & 2 & 2\\ 3 & -1 & -2\\ -2 & 0 & 2\end{pmatrix}=\begin{pmatrix}3 & -1 & -1\\ -\frac{3}{2} & \frac{1}{2} & 1\\ 1 & 0 & -1\end{pmatrix}.$$

Remark

Let A be a square matrix of order n. A is invertible if and only if the determinant of A is not zero. This is equivalent to saying that: A is singular if and only if the determinate of A is zero.

Exercise 4.10

Find the determinant, adjoint and inverse (if it exists) of each one the following matrices

a. $\begin{pmatrix}1 & 1\\ 2 & 3\end{pmatrix}$

b. $\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 4\\ 0 & 0 & 1\end{pmatrix}$

c. $\begin{pmatrix}1 & 2 & 1\\ 3 & 1 & 4\\ 5 & 1 & 5\end{pmatrix}$

4.6 Solutions of Systems of Linear Equations Using Cramer's Rule

We can use determinants to obtain another method, known as Cramer's Rule, to solve systems of n linear equations in n variables if the coefficient matrix of the system is invertible.

Consider a system of n linear equations AX = B, where A is a square matrix of order n. If A is invertible, then the system has a unique solution and this unique solution is given by

$X = A^{-1}B.$

Activity 4.11

Consider a system of two linear equations in two variables given by $\begin{cases} x + y = 1 \\ x - y = 3 \end{cases}$

a. Solve the given system.

b. Compute $\dfrac{\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}$ and $\dfrac{\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}$.

From your responses in Activity 4.11, observe that

a. $x = \dfrac{\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}$ and $y = \dfrac{\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}$;

b. The matrix $\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$ is obtained by replacing the first column of the coefficient matrix of the given system by $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and matrix $\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$ is obtained by replacing the second column of the coefficient matrix of the given system by $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

This method is called Cramer's Rule.

Theorem 4.7 (Cramer's Rule for Two Variables)

Given the system $\begin{cases} ax+by=e \\ cx+dy=f \end{cases}$ in two variables x and y, if $ad-bc\neq 0$, then the given system has a unique solution and it is given by $x=\dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{ad-bc}$ and $y=\dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{ad-bc}$.

Example 1

Use Cramer's rule, if possible, and solve the following system of linear equations:

$$\begin{cases} x+y=1 \\ 2x+y=5 \end{cases}$$

Solution

The given system in matrix form is given by:

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ 5 \end{pmatrix} \text{ and } \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}=1-2=-1\neq 0.$$

Thus, the system has a unique solution that is given by:

$$x=\frac{\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}=\frac{1-5}{-1}=\frac{-4}{-1}=4 \text{ and } y=\frac{\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}=\frac{5-2}{-1}=\frac{3}{-1}=-3.$$

Therefore, the solution set of the given system of linear equations is $\{(4,-3)\}$.

Exercise 4.11

Use Cramer's rule, if possible, and solve each of the following system of linear equations.

a. $\begin{cases} x+y=2 \\ 2x+3y=4 \end{cases}$

b. $\begin{cases} 2x+y=9 \\ 3x-4y=2 \end{cases}$

c. $\begin{cases} x+3y=8 \\ 2x-3y=4 \end{cases}$

d. $\begin{cases} -x+2y=0 \\ -2x+y=-1 \end{cases}$

Theorem 4.8 (Cramer's Rule for Three Variables)

Given the system

$$\begin{cases} a_{11}x+a_{12}y+a_{13}z=b_1 \\ a_{21}x+a_{22}y+a_{23}z=b_2 \\ a_{31}x+a_{32}y+a_{33}z=b_3 \end{cases}$$

in three variables x, y and z, if $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$, then the given system has a unique solution given by $x=\frac{D_x}{D}$, $y=\frac{D_y}{D}$ and $z=\frac{D_z}{D}$, where

$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, $D_x=\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$, $D_y=\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$ and

$D_z=\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$.

Example 2

Using Cramer's Rule (if possible) and solve the following system of linear equations.

$$\begin{cases} 2x+3y-5z=1 \\ x+y-z=2 \\ 2y+z=8 \end{cases}$$

Solution

The given system of linear equations is given in matrix form as:

$$\begin{pmatrix} 2 & 3 & -5 \\ 1 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix}.$$

Then the determinant of the coefficient matrix is:

$$D=\begin{vmatrix} 2 & 3 & -5 \\ 1 & 1 & -1 \\ 0 & 2 & 1 \end{vmatrix}=-7\neq 0.$$

Thus, the system has a unique solution and by Cramer's Rule:

$$x=\frac{\begin{vmatrix} 1 & 3 & -5 \\ 2 & 1 & -1 \\ 8 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -5 \\ 1 & 1 & -1 \\ 0 & 2 & 1 \end{vmatrix}}=\frac{-7}{-7}=1, \qquad y=\frac{\begin{vmatrix} 2 & 1 & -5 \\ 1 & 2 & -1 \\ 0 & 8 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -5 \\ 1 & 1 & -1 \\ 0 & 2 & 1 \end{vmatrix}}=\frac{-21}{-7}=3 \text{ and}$$

$$z=\frac{\begin{vmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 8 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -5 \\ 1 & 1 & -1 \\ 0 & 2 & 1 \end{vmatrix}}=\frac{-14}{-7}=2.$$

Therefore, the solution set of the given system of linear equations is $\{(1,3,2)\}$.

Remark

It is worth mentioning that, in finding the solution set of a system of n linear equations in n variables using **Cramer's Rule**, we have to evaluate $n + 1$ determinants. For systems with large number of equations, Gaussian Elimination Method is more efficient.

Exercise 4.12

Use Cramer's rule, if possible, and solve each of the following system of linear equations.

a. $\begin{cases} x+y+z=3 \\ -2x+2y+2z=5 \\ x+y+2z=4 \end{cases}$ b. $\begin{cases} x+y-z=1 \\ 2x+z=4 \\ y-z=5 \end{cases}$ c. $\begin{cases} x+y-z=1 \\ x-y+z=2 \\ 2x+2z=3 \end{cases}$

Systems of Linear Equations with no Solution or Infinitely Many Solutions

Given a system AX = B of n linear equations in n variables, where n is a positive integer, if $|A| = 0$, we cannot apply Cramer's rule to solve the system and further investigation is required.

If $|A| = 0$, then there are two possibilities: the system has no solution or the system has infinitely many solutions.

The cases of no solution or infinitely many solutions of a given system of linear equations are considered in the following two examples.

Example 3

Solve the following system of linear equations (if possible).

$$\begin{cases} -x+2y=3 \\ 3x-6y=0 \end{cases}$$

Solution

The coefficient matrix of the given system is $\begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix}$ and $\begin{vmatrix} -1 & 2 \\ 3 & -6 \end{vmatrix} = 6 - 6 = 0.$

This implies the system has no unique solution and we cannot use the Cramer's rule. The system has no solution or it has infinitely many solutions and we can use elimination method to determine.

The augmented matrix of the system is $\left(\begin{array}{cc|c} -1 & 2 & 3 \\ 3 & -6 & 0 \end{array}\right)$.

$$\left(\begin{array}{cc|c} -1 & 2 & 3 \\ 3 & -6 & 0 \end{array}\right) \xrightarrow{R_2 \to R_2 + 3R_1} \left(\begin{array}{cc|c} -1 & 2 & 3 \\ 0 & 0 & 9 \end{array}\right)$$

and the last matrix is in REF.

Thus, the given system is equivalent to the system:

$$\begin{cases} -x + 2y = 3 \\ 0 = 9 \end{cases}$$

The last equation, $0 = 9$, is always false.

Therefore, the system has no solution.

Graphing the lines represented by the two equations in Figure 4.1 reveals that the two lines are parallel.

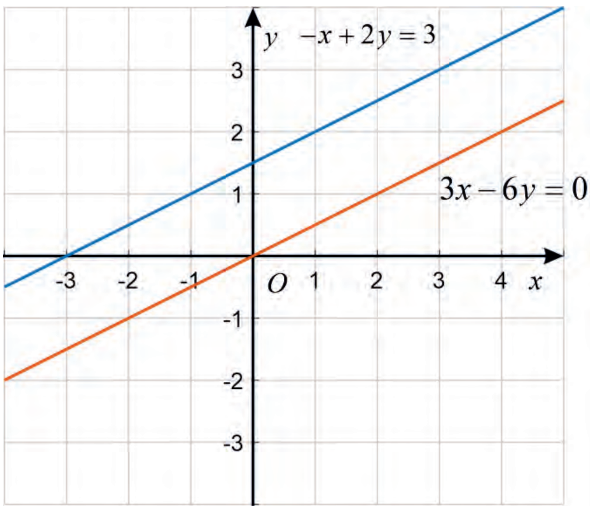


Figure 4.1

Example 4

Solve the following system linear equations.

$$\begin{cases} x-2y+3z=0 \\ 3x+y-2z=0 \\ 2x-4y+6z=0 \end{cases}$$

Solution

Let us first determine the determinant of the coefficient matrix.

The coefficient matrix of the system is $\begin{pmatrix} 1 & -2 & 3 \\ 3 & 1 & -2 \\ 2 & -4 & 6 \end{pmatrix}$ and its determinant is

$$\begin{vmatrix} 1 & -2 & 3 \\ 3 & 1 & -2 \\ 2 & -4 & 6 \end{vmatrix} = 0.$$

This implies the system has no unique solution and we cannot use the Cramer's rule. The system has no solution or it has infinitely many solutions and we can use elimination method to determine.

The augmented matrix of the system is:

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 3 & 1 & -2 & 0 \\ 2 & -4 & 6 & 0 \end{array}\right).$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 3 & 1 & -2 & 0 \\ 2 & -4 & 6 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 7 & -11 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and the last matrix is in Row Echelon Form.

Then, the given system of linear equations has the same solution as the system of linear equations:

$$\begin{cases} x-2y+3z=0 \\ 7y-11z=0 \end{cases}.$$

From the last equation, $7y - 11z = 0 \Rightarrow z = \frac{7}{11}y$.

Substituting this value in the first equation give us

$$x = 2y - 3z = 2y - 3\left(\frac{7}{11}y\right) = \frac{1}{11}y,\ y \in \mathbb{R}.$$

Therefore, the given system has infinitely many solutions given by:

$$\left\{\left(\frac{1}{11}y, y, \frac{7}{11}y\right) : y \in \mathbb{R}\right\}.$$

Exercise 4.13

Solve each of the following systems of linear equations, if possible.

a. $\begin{cases} x - y = 2 \\ -2x + 2y = -4 \end{cases}$

b. $\begin{cases} x + y - 2z = 3 \\ -2x - 2y + 4z = -6 \\ y - z = 5 \end{cases}$

c. $\begin{cases} x + y + z = 3 \\ -2x + 2y + 2z = 5 \\ x + y + 2z = 4 \end{cases}$

d. $\begin{cases} x + y - z = 1 \\ x - y + z = 2 \\ 2x + 2z = 3 \end{cases}$

4.7 Applications

Polynomial Interpolation

In studying a set of data that relates two variables x and y, it may be the case that we can use a polynomial to "fit" to the data. If such a polynomial can be established, it can be used to estimate values of x and y which have not been provided.

Example 1

Consider the data points $(0,1), (1,2)$ and $(2,22)$. Find an interpolating polynomial $p(x)$ of degree at most two, and estimate the value of $p(3)$.

Solution

The desired polynomial $p(x)$ is given by $p(x) = a + bx + cx^2$

Then, $p(0) = 1 = a$, $p(1) = 2 = a + b + c$ and $p(2) = 22 = a + 2b + 4c$.

This gives us a system of linear equations

$$\begin{cases} a = 1 \\ a + b + c = 2 \\ a + 2b + 4c = 22 \end{cases}$$

and the coefficient matrix of the given system is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \text{and} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = 1(4-2) = 2 \neq 0.$$

Then, the given system of linear equations has a unique solution.

We can solve the system using Cramer's rule as follows:

$$a = \frac{\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 22 & 2 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix}} = 1, \quad b = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 22 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix}} = -\frac{17}{2} = -8.5, \quad c = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 22 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix}} = \frac{19}{2} = 9.5.$$

Therefore, the required polynomial is:

$p(x) = 1 - 8.5x + 9.5x^2$ and $p(3) = 1 - 8.5(3) + 9.5(3)^2 = 61$.

Exercise 4.14

Consider data points $(0,2)$, $(1,5)$ and $(2,25)$. Find an interpolating polynomial $p(x)$ of at most degree two and estimate the values of $p(-1)$ and $p(3)$.

Area of a triangle in the *xy*-plane

Determinant can be used to find the area of a triangle whose three vertices are in the xy –plane.

The area of a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is in the plane is given by

$$\text{Area}(\Delta ABC) = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \text{ if } \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} > 0 \text{ and}$$

$$\text{Area}(\Delta ABC) = -\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \text{ if } \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} < 0.$$

Example 2

Find the area of the triangle with vertices $A(0,5)$, $B(0,0)$ and $C(5,0)$.

Solution

Using the cofactor expansion along the second row,

$$\begin{vmatrix} 0 & 5 & 1 \\ 0 & 0 & 1 \\ 5 & 0 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 0 & 5 \\ 5 & 0 \end{vmatrix} = 25 > 0.$$

Therefore, $\text{Area}(\Delta ABC) = \frac{1}{2}\begin{vmatrix} 0 & 5 & 1 \\ 0 & 0 & 1 \\ 5 & 0 & 1 \end{vmatrix} = \frac{1}{2}(25) = 12.5$ Square units.

Example 3

Find the area of the triangle with vertices $P(1,1)$, $Q(1,3)$ and $R(4,5)$

Solution

Using the cofactor expansion along the first row,

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 4 & 5 & 1 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 \\ 5 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 4 & 5 \end{vmatrix} = -2 + 3 - 7 = -6 < 0.$$

Therefore, Area $(\Delta PQR) = \frac{-1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 4 & 5 & 1 \end{vmatrix} = -\frac{1}{2}(-6) = 3$ Square units.

Exercise 4.15

Find the area of the triangle with vertices the following three points.

a. $(-2, 0)$, $(0, 2)$ and $(2, 0)$

b. $(1, 0)$, $(2, 1)$ and $(2,3)$

Test for collinear points in the *xy*-plane

Three points in xy −plane are said to be collinear if all the three points lie on the same line. Determinant can be used to determine whether three points in the xy −plane are collinear or not.

Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ in the xy-plane are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Example 4

Determine if the points $A(1,1)$, $B(2,2)$ and $C(3,3)$ are collinear.

Solution

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = 0.$$

This implies, the points $A(1,1)$, $B(2,2)$ and $C(3,3)$ are collinear.

Example 5

Determine if the points $A(0,1)$, $B(1,0)$ and $C(0,0)$ in the xy −plane are collinear.

Solution

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -1 \neq 0$$

This implies, the points $A(0,1)$, $B(1,0)$ and $C(0,0)$ are not collinear.

Two point form of equation of a line in the *xy*-plane

We can use determinant to find equation of a line in xy −plane passing through two points. An equation of a line passing through two distinct points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$

where (x, y) is a point on the line.

Example 6

Find an equation of the line passing through the points $P(2,0)$ and $R(0,3)$.

Solution

If (x, y) is a point on the line passing through $P(2,0)$ and $R(0,3)$, then:

$$\begin{vmatrix} x & y & 1 \\ 2 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix} = 0.$$

This implies $-2\begin{vmatrix} y & 1 \\ 3 & 1 \end{vmatrix} - \begin{vmatrix} x & y \\ 0 & 3 \end{vmatrix} = 0 \Rightarrow -2(y-3)-(3x-0)=0.$

Thus, the general equation of the line passing through the two points $P(2,0)$ and R $(0,3)$ is $3x+2y-6=0$.

Exercise 4.16

1. Determine if the following set of points are collinear.

   a. $(1,1)$, $(3,2)$ and $(4,3)$ b. $(1,0)$, $(2,1)$ and $(3,2)$

2. Find the general form of equation of the line passing through each of the following pairs of points.

   a. $(1,1)$ and $(3,2)$ b. $(1,-1)$ and $(-2,2)$ c. $(-1,1)$ and $(-4,2)$

Problem Solving

1. If the area of a triangle with vertices (–2, 0), (2, 0) and (0, k) is 8 sq. units, then determine the value(s) of k.

2. If the points $(3,k)$, $(k,1)$ and $(6,-1)$ are collinear, then determine the value(s) of k.

Summary

1. If $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then $\det(A)=a_{11}a_{22}-a_{12}a_{21}$.

2. If $A=(a_{ij})_{n\times n}$, then the matrix obtained by crossing out the i^{th} row and j^{th} column of A is denoted by A_{ij}.

   a. $\det(A_{ij})$ is called the minor of the element a_{ij} and

   b. $C_{ij}=(-1)^{i+j}\det(A_{ij})$ is called the cofactor of a_{ij}.

3. If $A=(a_{ij})_{3\times 3}$, $\det(A)=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}$, expansion of the determinant along the first row.

$$\text{If } A=\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix} \text{ or } A=\begin{pmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{pmatrix}, \text{ then } \det(A)=acf.$$

4. If A and B are two square matrices of order n, then,

   a. $\det(A)=\det(A^t)$

   b. $\det(AB)=\det(A)\det(B)$.

5. If A is an invertible matrix of order n, then det(A) is not equal to zero and the determinant of A^{-1} is given by

$$\det(A^{-1})=\frac{1}{\det(A)}$$

6. Let $A=(a_{ij})_{n\times n}$ be a square matrix of order n and $C_{ij}=(-1)^{i+j}\det(A_{ij})$ be the cofactor of the entry a_{ij} for $i=1,\ldots,n$ and $j=1,\ldots,n$.

$$\text{Then matrix } \operatorname{Adj}(A)=(C_{ij})^t_{n\times n}=\begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix} \text{ is called Adjoint of } A.$$

7. Let A be a square matrix of order n. Then $A\left(\mathrm{Adj}(A)\right)=\left(\mathrm{Adj}(A)\right)A=\det(A)I_n$ and if the determinant of A is not zero, then A is invertible and its inverse is given by

$$A^{-1}=\frac{1}{|A|}(\mathrm{Adj}(A)).$$

Review Exercise

1. Find the minor and the cofactor of each entry of the following matrices.

   a. $A=\begin{pmatrix}1 & 2\\ -3 & 5\end{pmatrix}$ b. $B=\begin{pmatrix}1 & 0 & -1\\ 2 & 3 & 1\\ -1 & 2 & 3\end{pmatrix}$

2. Find the determinant of each of the following matrices.

   a. $A=\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}$ b. $B=\begin{pmatrix}1 & 0 & 0\\ 2 & 3 & 0\\ 3 & -2 & -1\end{pmatrix}$ c. $C=\begin{pmatrix}2 & 1 & 3\\ 1 & -2 & 0\\ -1 & 2 & 4\end{pmatrix}$

   d. $D=\begin{pmatrix}-2 & 3 & -1\\ 0 & 1 & 2\\ 0 & 0 & 3\end{pmatrix}$ e. $E=\begin{pmatrix}-3 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{pmatrix}$ f. $F=\begin{pmatrix}1 & 2 & 3\\ 0 & 0 & 0\\ -1 & 3 & -2\end{pmatrix}$

3. Let A and B be square matrices of order 3 and $\det(A)=2$ and $\det(B)=-3$. Then find each of the following determinants.

   a. $\det(2AB)$ b. $\det(AB^t)$ c. $\det(A^{-1}B)$

4. Determine the invertiblity of each of the following matrices and find the inverse if it exists.

   a. $A=\begin{pmatrix}1 & 3\\ 0 & 2\end{pmatrix}$ b. $B=\begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & -2 & -1\end{pmatrix}$ c. $C=\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 2\\ 1 & 3 & 4\end{pmatrix}$

d. $D = \begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$ e. $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ f. $F = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix}$

5. Use Cramer's rule to solve each of the following systems of linear equations:

a. $\begin{cases} x + y = 1 \\ 2x + 3y = 0 \end{cases}$ b. $\begin{cases} x + 2y - z = 4 \\ 3x - y + z = 5 \\ 2x + 3y + 2z = 7 \end{cases}$

6. Find the area of the triangle with vertices the following three set of points.

a. $(-3, 0)$, $(0, 3)$ and $(1, 1)$ b. $(1, -1)$, $(1, 2)$ and $(3,2)$

7. Find the general form of equation of the line passing through each of the following pairs of points:

a. $(1, 1)$ and $(-1, 2)$ b. $(1, 2)$ and $(-2, -3)$ c. $(1, 0)$ and $(-4, 4)$

8. Determine if the following set of points are collinear:

b. $(1, 1)$, $(3, 2)$ and $(5, -2)$ b. $(1, 0)$, $(2, 2)$ and $(3, 3)$

UNIT 5

VECTORS

Unit Outcomes

By the end of this unit, you will be able to:

- Know basic concepts and procedures about vectors and operation on vectors.
- Know specific facts about vectors.
- Apply principles and theorem about vectors in solving problems involving vectors.

Unit Contents

KEY Words

- coordinate form of a vector
- cross product
- orthogonal vectors
- parallel vectors
- perpendicular vectors
- resolution of vectors
- scalar quantity
- Standard unit vector
- unit vector
- vectors quantity
- zero vector

Introduction

In Grade 9, you have discussed that measurement of any physical quantity can be expressed in terms of a number and a unit. For instance, physical quantities such as length, time, mass, electric current, area, volume, velocity, force, acceleration, work, etc. can be expressed in terms of a number and a unit.

Thus, physical quantities that are used to describe the motion of objects can be divided into two categories. The quantity is either a vector or a scalar. In this unit, you focus on various geometric and algebraic aspects of vector representation and vector algebra.

5.1 Revision on Vectors and Scalars

Activity 5.1

Classify the following physical quantities as scalar and vector quantities:

a. time	b. mass	c. work	d. torque
e. distance	f. displacement	g. length	h. velocity
i. force	j. volume	k. temperature	

From Activity 5.1, you have observed that there are physical quantities that can be expressed by magnitude and direction and there are physical quantities that can be expressed by a certain number associated with a suitable unit only. These physical quantities can be classified as a vector or a scalar physical quantity.

5.1.1 Scalars

Definition 5.1

Physical quantities expressed by a certain number associated with a suitable unit without any mention of direction in space is known as **scalar.** The number describing the quantity of a particular scalar is known as **its magnitude**. For example, time (15 sec), mass (20 kg), length (5 km), area (7 m^2), volume (10 m^3), temperature (37 °C).

5.1.2 Vectors

Definition 5.2

A vector is a physical quantity which can be expressed completely by stating both its magnitude with particular unit and direction. For example, car movement is usually described by giving the speed and the direction, say 60 km/h northeast. The car speed and car direction together form a vector quantity, velocity of the car.

Exercise 5.1

1. Make a list of physical quantities which are not mentioned in the introduction part, distinguish whether they are vectors or scalars.
2. Classify the following physical quantities as scalar and vector quantities.

   a. length b. velocity c. time d. force

5.2 Representation of Vectors

5.2.1 Components of Vectors

Activity 5.2

1. Discuss how a vector quantity can be represented in a coordinate form in a plane by giving examples.
2. If $\mathbf{v} = (1, 1)$ is a vector whose initial point is the origin as shown in Figure 5.1, then find a. the components of **v**. b. the direction of **v.**
3. If $\mathbf{u} = (-1, \sqrt{3})$ is a vector whose initial point is the origin as shown in Figure 5.1, then find a. the components of **v.** b. the direction of **v.**

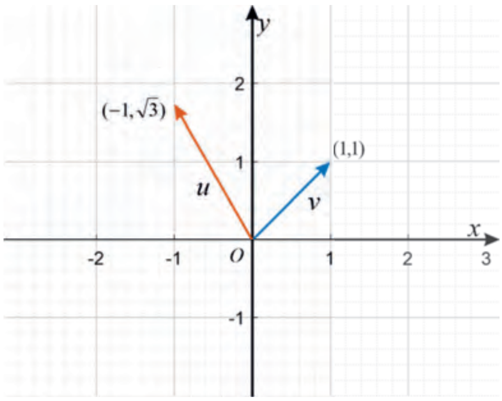


Figure 5.1

From Activity 5.2, you have observed that how vectors can be represented in a coordinate or a column form in a plane.

Definition 5.3

1. If **v** is a vector in the plane whose initial point is the origin, O (0, 0) and whose terminal point is Q (x, y) then the coordinate form of a vector **v** = vector OQ = (x, y) or the column form of a vector **v** = $\binom{x}{y}$ as shown in Figure 5.2. The numbers x and y are called components of a vector **v**, the direction of the vector can be given by the slope m = $\frac{y}{x}$.
2. If **v** is a vector in the plane whose initial point P (x_1, y_1) and whose terminal point Q(x_2, y_2) the coordinate form of a vector **v** = vector PQ = $(x_2 - x_1, y_2 - y_1)$ or the column form of a vector **v** $= \binom{x_2 - x_1}{y_2 - y_1}$ as shown in Figure 5.3.

   The numbers $x_2 - x_1$ and $y_2 - y_1$ are called components of a vector **v**, the direction of the vector **v** can be given by the inclination of the line with slope $m = \frac{y_2 - y_1}{x_2 - x_1}$.

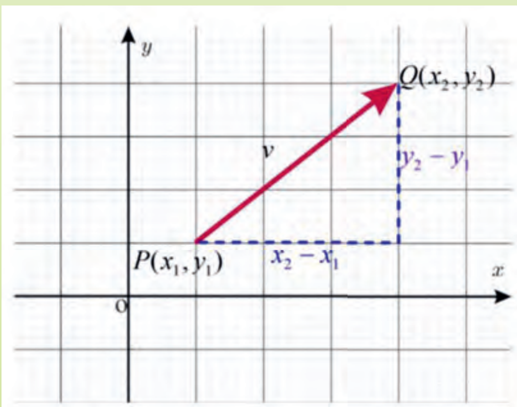


Figure 5.2

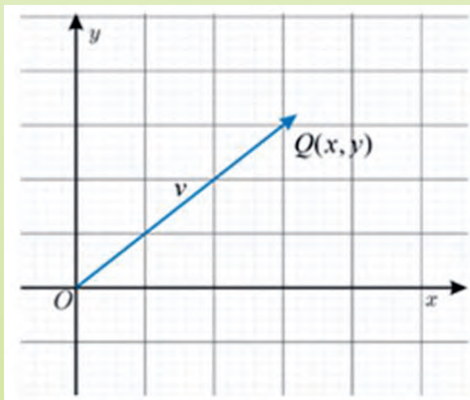


Figure 5.3

Note

1. If both the initial and terminal points lie at the origin, then **v** is the **zero vector** and is given by **v** = (0, 0) or **v** = $\binom{0}{0}$.

Example 1

Find the coordinate and column forms of the vector **v** that has initial point (0, 0) and terminal point (3, 6).

Solution

Let O = (0, 0) and Q = (3, 6), as shown Figure 5.4.

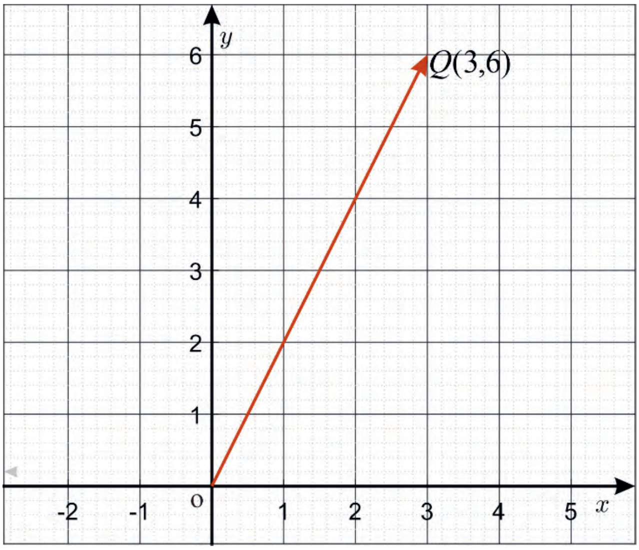


Figure 5.4

Accordingly, coordinate form of vector **v** is **v** = (3, 6) and the column form of vector **v** is $\mathbf{v} = \binom{3}{6}$.

Example 2

Find the coordinate and column forms of the vector **v** that has initial point (1, 2) and terminal point (6, 7).

Solution

Let P = (1, 2) and Q = (6, 7), as shown in Figure 5.4.

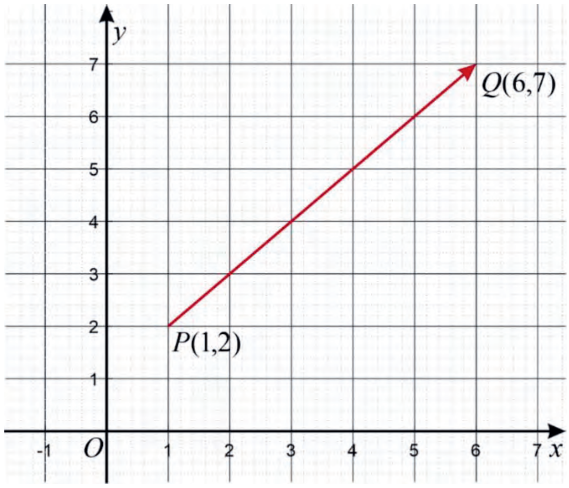


Figure 5.5

Thus, the coordinate form of vector **v** is

v = $\overrightarrow{PQ} = (x_2 - x_1, y_2 - y_1) = (6 - 1, 7 - 2) = (5, 5)$ and the column form of vector **v** is **v** = $\binom{5}{5}$.

Exercise 5.2

1. Find the coordinate and column forms of the vector **v** that has initial point (0, 0) and terminal point (7, 8).
2. Find the coordinate and column forms of the vector **v** that has initial point (2, 5) and terminal point (5, 10).
3. Find the coordinate and column forms of the vector **v** that has initial point A (a_1, a_2) and terminal point B (b_1, b_2).

5.2.2 Addition and Subtraction of Vectors

a. Addition of Vectors

Activity 5.3

1. What do you know about addition of vectors?
2. Consider vector **u** as a displacement that Kebede walks 5 m due east and then **v** as a displacement that Kebede walks 6 m due north. Find these two displacements as a single displacement.

From Activity 5.3, you can add two vectors geometrically by joining tail to head of the resultant vector **r** = **u** + **v**, which describes addition of vectors and can be found by placing **v** such that its tail is at the same point as the head of **u**. The resultant vector **r**, or addition of vectors **u** + **v**, has its tail at the tail of **u** and its head at the head of **v**, as shown in Figure 5.6, indicating triangular law of vector addition.

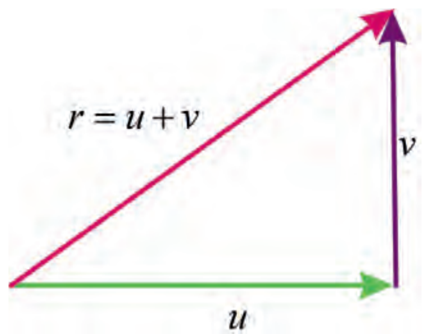


Figure 5.6

Definition 5.4 **(Parallelogram method of addition of vectors (head-to-tail method))**

If **u** and **v** are two vectors, the sum **u** + **v** is the vector determined by translating vector **v** until its tail coincides with the head of **u**. Then, the directed line segment from the tail of **u** to the head of **v** is the vector **u** + **v**.

Theorem 5.1 (Commutative property of vector addition)

If **u** and **v** are any two vectors, then **u** + **v** = **v** + **u.**

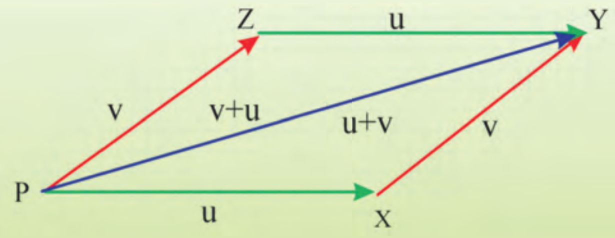


Figure 5.7

Theorem 5.2 (Associative Property of vector Addition)

If **u, v** and **w** are any three vectors, then (**u** + **v**) + **w** = **u** + (**v** + **w**).

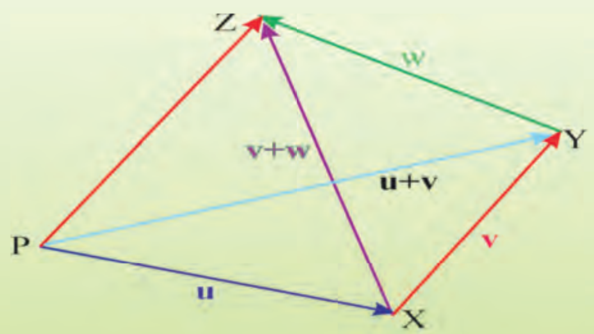


Figure 5.8

Example 1

If **u** = (2, 5), **v** = (7, 3) and **w** = (1, 6) as shown in figure 5.9, then find

a. **u** + **v** b. **v** + **u**

c. (**u** + **v**) + **w** d. **u** + (**v** + **w**)

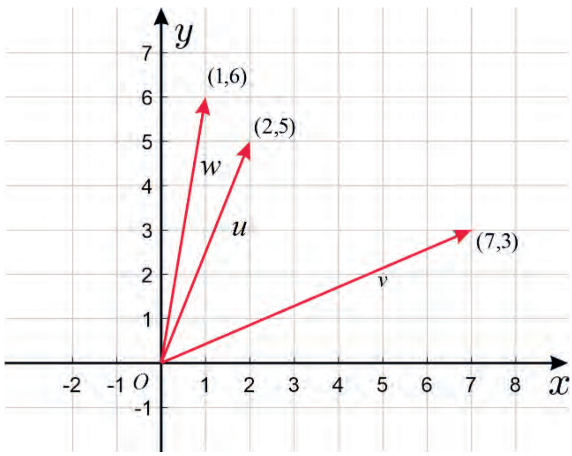


Figure 5.9

Solution

Substitute the given values of **u**, **v** and **w** into the definition of vector addition.

a. $\mathbf{u} + \mathbf{v} = (2, 5) + (7, 3) = (2 + 7, 5 + 3) = (9, 8)$.

b. $\mathbf{v} + \mathbf{u} = (7, 3) + (2, 5) = (7 + 2, 3 + 5) = (9, 8)$.

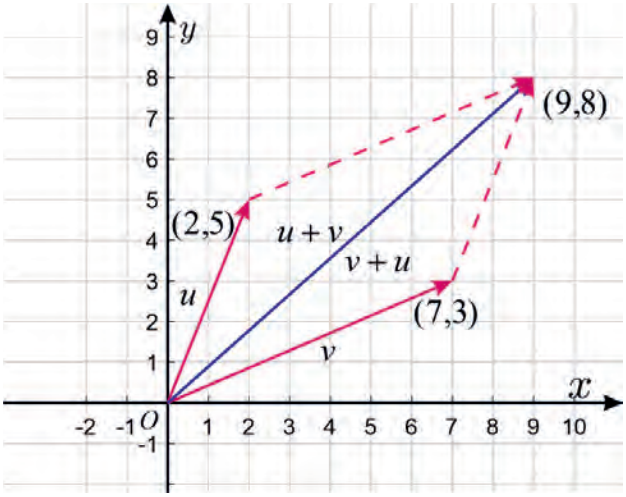


Figure 5.10

c. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = ((2, 5) + (7, 3)) + (1, 6) = (2 + 7, 5 + 3) + (1, 6) = (9, 8) + (1, 6)$

$= (9 + 1, 8 + 6) = (10, 14)$.

d. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (2, 5) + ((7, 3) + (1, 6)) = (2, 5) + (7 + 1, 3 + 6) = (2, 5) + (8, 9)$

$= (2 + 8, 5 + 9) = (10, 14)$.

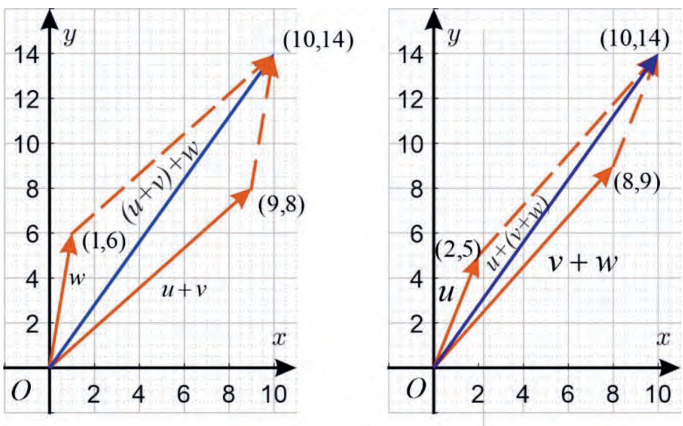


Figure 5.11

Note

1. For any three vectors **u**, **v** and **w**, if **u** = **v** and **v** = **w**, then **u** = **w**.
2. The zero vector **0** has the following property:

   For any vector **u**, **u** + **0** = **0** + **u** = **u**.
3. If **u** = (x_1, y_1) and **v** = (x_2, y_2) then **u** + **v** = $(x_1 + x_2, y_1 + y_2)$.

b. Subtraction of Vectors

Activity 5.4

Suppose that **u** and **v** are two vectors. How can you interpret the subtraction of these vectors in terms of addition?

From activity 5.4, you have observed that **u** – **v** is equal to **u** + (−**v**), that is the vector sum of **u** and −**v**. Now, you can reverse vector **v** and then add using the parallelogram method of addition of vectors.

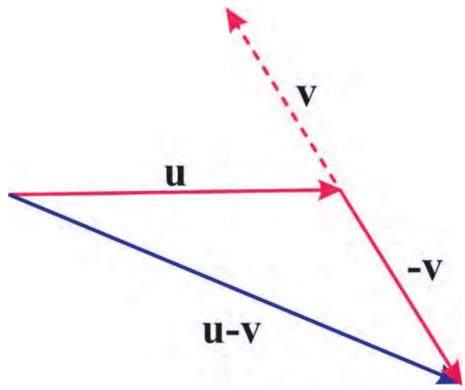


Figure 5.12

This figure shows subtraction of vector **u** – **v** as addition of **u** and (−**v**) using the parallelogram method of addition of vectors.

Moreover, you can also subtract these vectors using the triangular law of vector addition as follows. Draw the position **u** and **v** so that their initial points coincide. Here, the vector from the terminal point of **v** to the terminal point of **u** is then the vector **u** – **v** as shown in Figure 5.13.

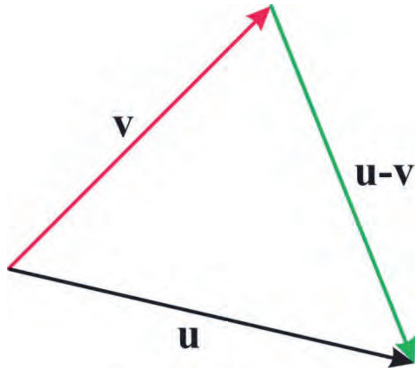


Figure 5.13

Note

1. Subtraction of two vectors **u** and **v** is adding the negative of vector **v** to the vector **u**. That means **u** – **v** = **u** + **(−v).**
2. You can do subtraction of vectors just like how you can do subtraction of scalers. You subtract the corresponding components of vectors while subtracting vectors.
3. If **v** is any non-zero vector and –**v** is the negative of **v**, then **v** + (–**v**) = 0.
4. The subtraction of vectors is not commutative.
5. The subtraction of vectors is not associative.

Example 2

If **u** = (3, 6), **v** = (8, 4) and **w** = (2, 7), then find

a. **u** − **v** b. **v** − **u** c. (**u** − **v**) − **w** d. **u** − (**v** − **w**)

Solution

Substitute the given values of **u**, **v** and **w**.

a. **u** − **v** = **u** + (− **v**) = (3, 6) + (−(8, 4)) = (3, 6) + (−8, −4)

= (3 + (−8), 6 + (−4)) = (−5, 2) .

b. **v** − **u** = (8, 4) − (3, 6) = (8 − 3, 4 − 6) = (5, −2).

c. $(\mathbf{u} - \mathbf{v}) - \mathbf{w} = ((3, 6) - (8, 4)) - (2, 7) = (3 - 8, 6 - 4) - (2, 7)$

$= (-5, 2) - (2, 7) = (-5 - 2, 2 - 7) = (-7, -5).$

d. $\mathbf{u} - (\mathbf{v} - \mathbf{w}) = (3, 6) - ((8, 4) - (2, 7)) = (3, 6) - (8 - 2, 4 - 7)$

$= (3, 6) - (6, -3) = (3 - 6, 6 - (-3)) = (-3, 9).$

Exercise 5.3

1. Find a. $\mathbf{u} + \mathbf{v}$ b. $\mathbf{u} - \mathbf{v}$ c. $(\mathbf{u} + \mathbf{v}) + \mathbf{w}$ d. $(\mathbf{u} - \mathbf{v}) - \mathbf{w}$

   if $\mathbf{u} = (3, 4)$, $\mathbf{v} = (5, -1)$ and $\mathbf{w} = (2, -5)$.

2. Write any point A, B, C and F in your notebook and draw diagrams to illustrate the following vector equations:

   a. $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ b. $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CF} = \overrightarrow{AF}$ c. $\overrightarrow{AB} - \overrightarrow{BC} = \overrightarrow{AC}$

3. Graphically add the given vectors **u** and **v** shown in Figure 5.14 by using head to tail rule

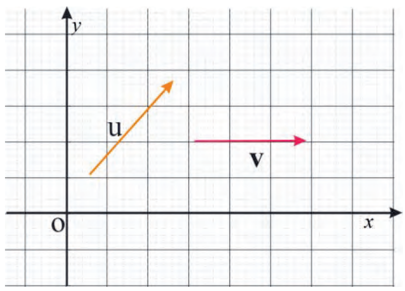


Figure 5.14

5.2.3 Multiplication of Vectors by Scalars

Activity 5.5

Let $\mathbf{u} = (2, 1)$. Then graph the pair of vectors on the same xy-plane.

a. $\mathbf{u}$ and $\mathbf{w}$, where $\mathbf{w} = \mathbf{u} + \mathbf{u} + \mathbf{u}$

b. $\mathbf{u}$ and $-\mathbf{u}$

c. $\mathbf{u}$ and $-2\mathbf{w}$, where $\mathbf{w} = \mathbf{u} + \mathbf{u} + \mathbf{u}$

Suppose you have a vector **u**, then if this vector is multiplied by a scalar quantity

$k = -1, 2, 3, -3, \frac{-1}{2}$ and so on, you will get the value as shown in Figure 5.15.

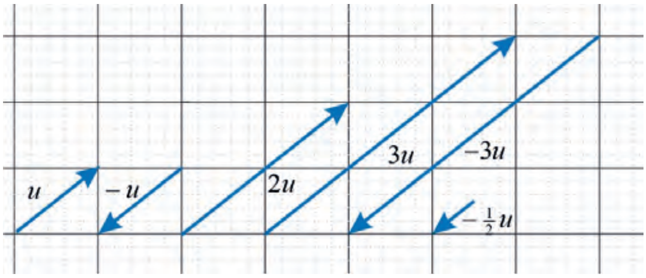


Figure. 5.15

Hence, from the above given set of vectors, you have observed that the direction of vector **u** remains the same when the value of the scalar is positive and the direction becomes exactly opposite when the value of the scalar is negative. In both cases the magnitude keeps changing depending upon the values of the scalar multiple.

Note

The product $k\mathbf{v}$ of a vector **v** and a scalar k is a vector with a magnitude that is $|k|$ times the magnitude of **v**, and with a direction that is the same as the direction of **v** if $k > 0$, and opposite the direction of **v** if $k < 0$. This is called scalar multiplication. If $k = 0$ or $\mathbf{v} = \mathbf{0}$, then $k\mathbf{v} = \mathbf{0}$.

Example 1

Let $\mathbf{u} = (-1, 3)$. Find $7\mathbf{u}$.

Solution

$7\mathbf{u} = 7(-1, 3) = (7(-1), 7(3)) = (-7, 21)$.

Theorem 5.3 (Properties of Scalar Multiplication)

Let **u** and **v** be vectors, let k_1 and k_2 be scalars. Then the following properties are true.

1	The magnitude of the vector which is multiplied by a scalar is equal to the absolute value of the scalar times the magnitude of the vector.	$\lvert k_1\mathbf{u}\rvert = \lvert k_1\rvert\lvert\mathbf{u}\rvert$
2	Associative Property	$k_1(k_2\,\mathbf{u}) = (k_1k_2)\,\mathbf{u}$
3	Distributive Property	i. $(k_1 + k_2)\,\mathbf{u} = k_1\mathbf{u} + k_2\mathbf{u}$ ii. $k_1(\mathbf{u} + \mathbf{v}) = k_1\mathbf{u} + k_1\mathbf{v}$
4	Identity Property	$1 \cdot \mathbf{u} = \mathbf{u}$
5	Multiplicative Property of 0	$0 \cdot \mathbf{u} = 0$

Example 2

If $\mathbf{u} = (5, -9)$ and $\mathbf{v} = (3, 5)$, then find

a. $2(3\mathbf{u})$ **b.** $(3 + 2)\,\mathbf{u}$ **c.** $3(\mathbf{u} + \mathbf{v})$ **d.** $1 \cdot \mathbf{v}$ **e.** $0 \cdot \mathbf{v}$

Solution

a. $2(3\mathbf{u}) = 2(3(5, -9)) = 2((3(5), 3(-9))) = 2(15, -27) = (30, -54)$

b. $(3 + 2)\,\mathbf{u} = (3 + 2)\,(5, -9) = 5(5, -9) = (25, -45)$

c. $3(\mathbf{u} + \mathbf{v}) = 3((5, -9) + (3, 5))$

$= 3(5, -9) + 3\,(3, 5) = (15, -27) + (9 + 15)$

$= (24, -12)$

d. $1 \cdot \mathbf{v} = 1 \cdot (3, 5) = (3, 5)$

e. $0 \cdot \mathbf{v} = 0 \cdot (3, 5) = (0, 0)$

Exercise 5.4

1. Let **u** = (11, 16) and **v** = (–8, 12). Find

   a. 5**u** b. 3(4**u**) c. (5 + 3) **u** d. 4(**u** + **v**) e. 1· **v** f. 0 · **v**

2. In the triangle ABC, $\overrightarrow{AB}$ represents **u** and $\overrightarrow{BC}$ represents **v.** If D is the midpoint of AB, then express $\overrightarrow{AC}$, $\overrightarrow{CA}$ and $\overrightarrow{DC}$ in terms of **u** and **v.**

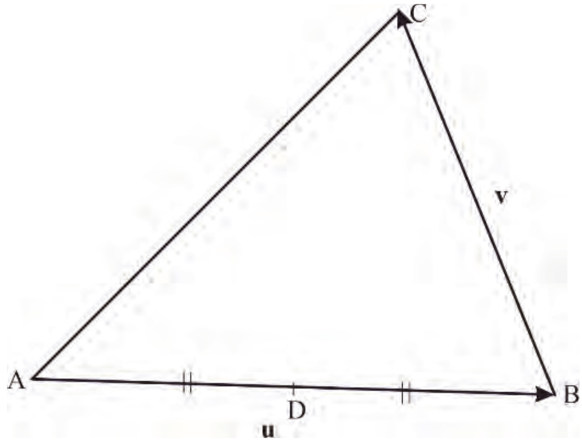


Figure 5. 16

3. ABCD is a parallelogram. $\overrightarrow{AB}$ represents **u** and $\overrightarrow{BC}$ represents **v.** If M is the midpoint of AC, and N is the midpoint of BD, find $\overrightarrow{BM}$ and $\overrightarrow{AN}$ in terms of **u** and **v,** afterwards show that M and N are coincident.

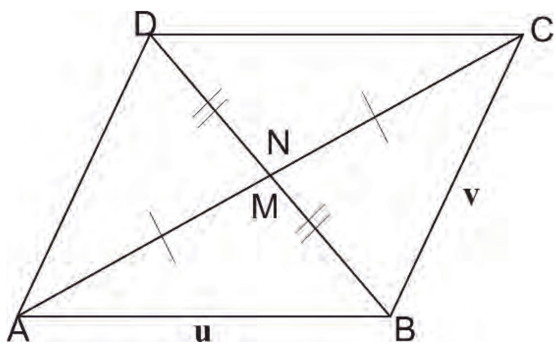


Figure 5. 17

5.2.4 Unit Vectors

Activity 5.6

For vectors in the plane, using standard unit vector " **i** " and "**j**" as shown in Figure 5.18, express a, b, c, d and e in terms of **i** and **j**.

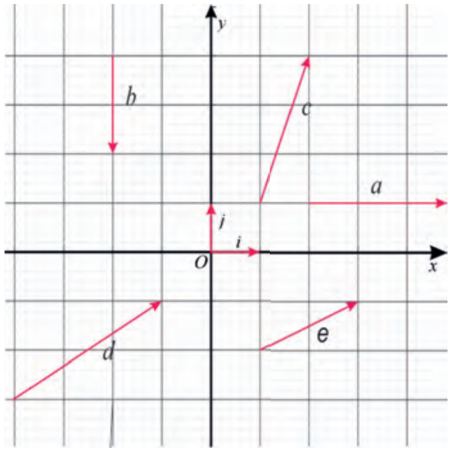


Figure 5.18

From Activity 5.6, you should have observed that, for example, $\mathbf{c} = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{e} = 2\mathbf{i} + \mathbf{j}$. Then, the vector $\mathbf{c} + \mathbf{e}$ is simply $3\mathbf{i} + 4\mathbf{j}$. Adding vectors in terms of **i** and **j** is just a matter of adding components.

Consider a vector **u** whose initial point is the origin O (0, 0) and whose terminal point is the point P (a, b) as shown in Figure 5.19.

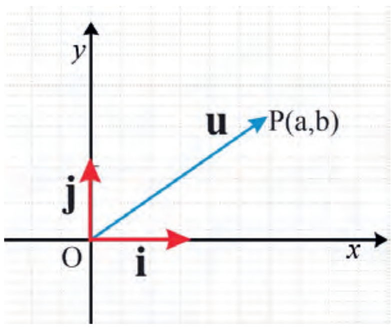


Figure 5.19

Hence, if a vector $\mathbf{u} = (a, b)$, then $\mathbf{u} = (a, b) = (a, 0) + (0, b) = a\,(1, 0) + b\,(0, 1)$

$= a\mathbf{i} + b\mathbf{j}$ where the unit vectors are $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$.

Example 1

Express the following vectors in terms of the standard unit vectors **i** and **j**:

a. $(5, -6)$ b. $(-4, 8)$ c. $(-7, 0)$ d. $(0, 3)$

Solution

a. $(5, -6) = (5, 0) + (0, -6) = 5\,(1, 0) + (-6)\,(0, 1) = 5\mathbf{i} + (-6)\,\mathbf{j} = 5\mathbf{i} - 6\mathbf{j}$.

b. $(-4, 8) = (-4, 0) + (0, 8) = -4\,(1, 0) + 8\,(0, 1) = -4\mathbf{i} + 8\mathbf{j}$.

c. $(-7, 0) = (-7, 0) + (0, 0) = -7\,(1, 0) + 0\,(0, 1) = -7\mathbf{i} + 0\mathbf{j} = -7\mathbf{i}$.

d. $(0, 3) = (0, 0) + (0, 3) = 0\,(1, 0) + 3(0, 1) = 0\mathbf{i} + 3\mathbf{j} = 3\mathbf{j}$.

Example 2

Express each of the following vectors as a vector in a coordinate:

a. $\mathbf{i} + 4\mathbf{j}$ b. $3\mathbf{i} - 3\mathbf{j}$ c. $-7\mathbf{i} + 8\mathbf{j}$

Solution

a. $\mathbf{i} + 4\mathbf{j} = (1, 0) + 4(0, 1) = (1, 0) + (0, 4) = (1 + 0, 0 + 4) = (1, 4)$.

b. $3\mathbf{i} - 3\,\mathbf{j} = 3(1, 0) + -3(0, 1) = (3, 0) + (0, -3) = (3 + 0, 0 + (-3)) = (3, -3)$.

c. $-7\mathbf{i} + 8\mathbf{j} = -7(1, 0) + 8(0, 1) = (-7, 0) + (0, 8) = (-7 + 0, 0 + 8) = (-7, 8)$.

Exercise 5.5

1. Express the vector $(3, 4)$ in terms of the standard unit vectors **i** and **j**.
2. Express $\overrightarrow{PQ}$ in terms of unit vectors **i** and **j**, when the end points are P $(6, 2)$ and Q $(1, 3)$.
3. Let **u** be the vector with initial point $(-4, 6)$ and terminal point $(-10, 5)$. Write **u** in terms of the standard unit vectors **i** and **j**. Also, represent it graphically.
4. Write the following vectors as a vector in the coordinate:

a. $\mathbf{i} + \mathbf{j}$ b. $-2\mathbf{i} + \frac{1}{5}\mathbf{j}$ c. $-3\mathbf{i} + (-5\mathbf{j})$ d. $\frac{3}{2}\mathbf{i} - 6\mathbf{j}$

5.2.5 Norm of Vectors

If $\overrightarrow{AB}$ is a vector with initial point (x_1, y_1) and terminal point (x_2, y_2) as shown in Figure 5.20, then its position vector **u** is determined as

$\mathbf{u} = (x_2 - x_1, y_2 - y_1) = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$

Thus, $(x_2 - x_1)$ and $(y_2 - y_1)$ are the coordinates of **u** with respect to the base (**i**, **j**).

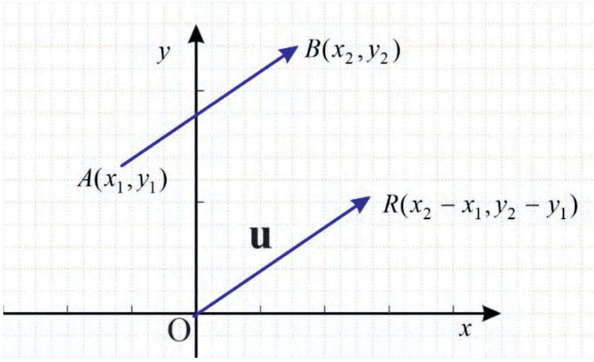


Figure 5.20

The norm or the magnitude of a vector $\mathbf{u} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$ is given by

$|\mathbf{u}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example 1

Find the norms of the following vectors:

a. $(3, -3)$ b. $(4, 6)$ c. $(-4, 0)$ d. $(0, 6)$

Solution

a. The norm or magnitude of vector $\mathbf{u} = (3, -3)$ is

$|\mathbf{u}| = \sqrt{3^2 + (-3)^2} = \sqrt{9 + 9} = 3\sqrt{2}$.

b. The norm or magnitude of vector $\mathbf{u} = (4, 6)$ is

$|\mathbf{u}| = \sqrt{4^2 + 6^2} = \sqrt{16 + 36} = 2\sqrt{13}$.

c. The norm or magnitude of vector $\mathbf{u} = (-4, 0)$ is

$|\mathbf{u}| = \sqrt{(-4)^2 + 0^2} = \sqrt{16 + 0} = 4$.

d. The norm or magnitude of vector $\mathbf{u} = (0, 6)$ is $|\mathbf{u}| = \sqrt{0^2 + (6)^2} = \sqrt{0 + 36} = 6$.

Definition 5.5

Any vector that has magnitude which equals to 1 is called unit vectors and denoted by $\hat{\mathbf{v}}$ (read as **v** cap). Unit vectors are generally used to denote the direction of a vector.

- If **u** is any non -zero vector then the unit vector in the direction of **u** is given as $\hat{\mathbf{u}} = \mathbf{u} \times \frac{1}{|\mathbf{u}|}$.
- For vectors in two dimensions, in general $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$ where x and y are scalars and its magnitude is given by the length or norms of vector OA.

  Where OA $= \sqrt{x^2 + y^2}$

  $\Rightarrow |\mathbf{u}| = \sqrt{x^2 + y^2}$

  In this case, we say that **u** is expressed as a linear combination of **i** and **j**.

Example 2

If **u** = (1, 2) and **v** = (3, 4), then find:

a. the unit vector in the direction of **v**.

b. the unit vector in the direction of **u** + **v**.

c. the unit vector in the direction of **u** – **v**.

Solution

a. $|\mathbf{v}| = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

Substitute the given values of **v** and |**v**| into $\hat{\mathbf{v}} = \mathbf{v} \times \frac{1}{|\mathbf{v}|}$

Hence, $\hat{\mathbf{v}} = \mathbf{v} \times \frac{1}{|\mathbf{v}|}$

$\hat{\mathbf{v}} = (3, 4) \times \frac{1}{5} = (\frac{3}{5}, \frac{4}{5})$.

Thus, the unit vector in the direction of **v** is $(\frac{3}{5}, \frac{4}{5}) = \frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{j}$.

b. $\mathbf{u} + \mathbf{v} = (1, 2) + (3,4) = (4,6)$

The norm of $|\mathbf{u} + \mathbf{v}| = \sqrt{x^2 + y^2} = \sqrt{4^2 + 6^2} = \sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13}$

Substitute the given values of $\mathbf{u}+\mathbf{v}$ and $|\mathbf{u}+\mathbf{v}|$ into $\widehat{(\mathbf{u}+\mathbf{v})} = (\mathbf{u}+\mathbf{v}) \times \frac{1}{|\mathbf{u}+\mathbf{v}|}$

Hence, $\widehat{(\mathbf{u}+\mathbf{v})} = (\mathbf{u}+\mathbf{v}) \times \frac{1}{|\mathbf{u}+\mathbf{v}|}$

$$= (4, 6) \times \frac{1}{2\sqrt{13}} = \left(\frac{4}{2\sqrt{13}}, \frac{6}{2\sqrt{13}}\right) = \left(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right) = \left(\frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}\right)$$

Thus, the unit vector in the direction of $\mathbf{u}+\mathbf{v}$ is $\left(\frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}\right) = \frac{2\sqrt{13}}{13}\mathbf{i} + \frac{3\sqrt{13}}{13}\mathbf{j}$.

c. $\mathbf{u}-\mathbf{v} = (1, 2) - (3, 4) = (-2, -2)$.

The norm of $|\mathbf{u}-\mathbf{v}| = \sqrt{x^2+y^2} = \sqrt{(-2)^2+(-2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$.

Substitute the given values of $\mathbf{u}-\mathbf{v}$ and $|\mathbf{u}-\mathbf{v}|$ into $\widehat{(\mathbf{u}-\mathbf{v})} = (\mathbf{u}-\mathbf{v}) \times \frac{1}{|\mathbf{u}-\mathbf{v}|}$.

Hence, $\widehat{(\mathbf{u}-\mathbf{v})} = (\mathbf{u}-\mathbf{v}) \times \frac{1}{|\mathbf{u}-\mathbf{v}|} = (-2, -2) \times \frac{1}{2\sqrt{2}} = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

Thus, the unit vector in the direction of $\mathbf{u}-\mathbf{v}$ is $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$.

Exercise 5.6

1. Find the norm (or magnitude) of each of the following vectors.

   a. $\mathbf{u} = (1, 1)$ b. $\mathbf{v} = (-2, 1)$ c. $\mathbf{u} = (5, 0)$

   d. $\mathbf{v} = \left(\frac{1}{4}, \frac{3}{2}\right)$ e. $\mathbf{u} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{5}\right)$

2. If $\mathbf{u} = 3\mathbf{i} + 5\mathbf{j}$ and $\mathbf{v} = 7\mathbf{i} - \mathbf{j}$, then find

   a. $|\mathbf{u}+\mathbf{v}|$ b. $|\mathbf{u}-\mathbf{v}|$ c. $r(\mathbf{u}+2\mathbf{v}), r \in \mathbb{R}$ d. $\left|\frac{1}{4}\mathbf{u}-\mathbf{v}\right|$

3. Find a unit vector in the direction of the vector (3, 6).

4. Find a unit vector in the direction opposite to the vector (1, 3).

5. Find two, unit vectors, one in the same direction and the other opposite to the vector $\mathbf{u} = (a, b) \neq 0$.

Definition 5.6

Two vectors **u** and **v** are parallel **if u** = k**v** for some non-zero k. furthermore**,** if $k > 0$ they are actually in the same direction, if $k < 0$, they are in opposite direction.

Note

1. Two vectors **u** = u_1**i** + u_2**j** and **v** = v_1**i** + v_2**j** are equal if and only if their components are equal. i.e., $u_1 = v_1$ and $u_2 = v_2$.
2. Two or more vectors are said to be equal if they have the same magnitude and the same direction.

Example 3

Determine which of the vectors are equal to each other as shown in Figure 5.21.

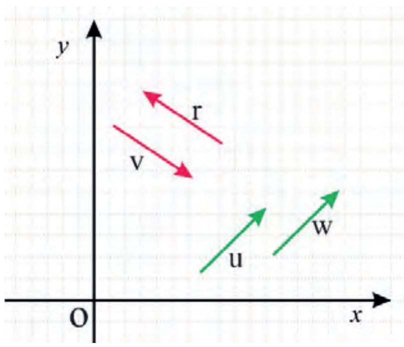


Figure 5.21

Solution

You will compare the given vectors to determine their magnitudes and directions.

First, you determine the magnitudes of the vectors.

It is obvious that |**u**| = |**w**| and |**v**| = |**r**|. The magnitudes of the vectors **u** and **w** as well as the magnitude of vectors **v** and **r** are the same.

To compare the directions, it can be observed that the vectors **u** and **w** are parallel to each other with arrows on the same direction whereas the vectors **v** and **r** are parallel to each other with arrows on the opposite direction.

Thus, we can conclude that:

1. If two vectors have the same magnitude and direction, they are equal.
2. If two vectors have the same magnitude but opposite directions, one is inverse vector to the other.

Example 4

If $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$, which of the vectors below are parallel to **u** or **v**?

1. $a = -2\mathbf{i} - \mathbf{j}$
2. $b = 5\mathbf{i} - 10\mathbf{j}$
3. $c = 4\mathbf{i} - 8\mathbf{j}$
4. $d = 6\mathbf{i} + 3\mathbf{j}$
5. $e = -2\mathbf{i} + 4\mathbf{j}$
6. $f = -4\mathbf{i} - 2\mathbf{j}$

Solution

The given vectors are expressed in terms of the unit vectors i and j. To determine whether or not they are parallel, you can check if their respective components can be expressed as scalar multiples of each other or not. Therefore, vectors **a, d** and **f** are parallel to vector **u** whereas vectors **b, c** and **e** are parallel to vector **v.**

Exercise 5.7

1. Determine whether the following two vectors are parallel or not. Then determine the magnitude of the two vectors.

   a. $\mathbf{u} = (3\mathbf{i} + 4\mathbf{j})$ and $\mathbf{v} = (9\mathbf{i} + 12\mathbf{j})$

   b. $\mathbf{w} = (6\mathbf{i} + 8\mathbf{j})$ and $\mathbf{u} = -3\mathbf{i} - 4\mathbf{j}$

   c. $\mathbf{v} = (\frac{1}{4}, \frac{3}{2})$ and $\mathbf{u} = (\frac{1}{2}, \frac{3}{4})$

2. The column vectors **u** and **v** are defined by $\mathbf{u} = \begin{pmatrix} 8 - x \\ 6 - y \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} x - 4 \\ y + 2 \end{pmatrix}$.

   Given that $\mathbf{u} = \mathbf{v}$

   a. Find the values of x and y.

   b. Find the values of $|\mathbf{u}|$ and $|\mathbf{v}|$.

5.3 Vector Product

The products of vectors are discussed in this sub-unit as follows:

5.3.1 Scalar (inner or dot) Product

Activity 5.7

1. Find the angle between the following pairs of vectors.

   a. $4\mathbf{i}$ and $2\mathbf{j}$ b. $\mathbf{i}$ and $3\mathbf{i}$ c. $2\mathbf{i}$ and $-\mathbf{i}$ d. $\mathbf{i}+\mathbf{j}$ and $3\mathbf{j}$ e. $\mathbf{i}$ and $\mathbf{i}+\sqrt{3}\mathbf{j}$

2. Consider two vectors such as **u** and **v** both of which have the same initial point. The angle formed by these two vectors is θ as shown in figure 5.22. Discuss how you can express θ in terms of $|\mathbf{u}|$ and $|\mathbf{v}|$.

In previous section, you have seen three vector operations: vector addition, subtraction and multiplication by a scalar, each of which provides another **vector.** In this section, you will study another vector operation like the inner or dot product. This product provides a **scalar,** rather than **a vector.**

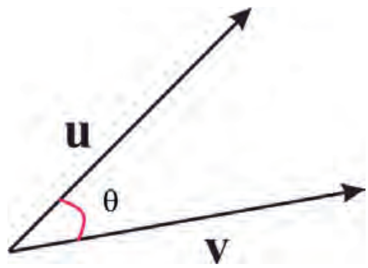


Figure 5.22

As shown in Figure. 5.22, the tails of the two vectors coincide and the angle between the vectors has been labelled θ**.** It is expressed θ in terms of $|\mathbf{u}|$ and $|\mathbf{v}|$ as $\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}||\mathbf{v}|\cos\theta$ which defines the scalar product of two vectors such as **u** and **v**. It is very important to use the dot in the formula. The dot is the symbol for the **scalar product**, and is the reason why the scalar product is also known as **the dot product.**

The angle θ is always chosen to lie between 0 and π, and the tails of the two vectors must coincide.

Theorem 5.4

If **u** and **v** are vectors and θ is the angle between **u** and **v,** then the dot product of **u** and **v**, denoted by **u** ▪ **v** is given by **u** ▪ **v** = |**u**||**v**| cosθ

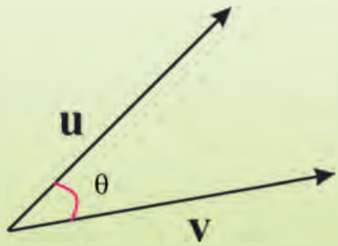


Figure. 5.23

Example 1

Suppose the vector **u** has norm 8 and the vector **v** has norm 7. Suppose also that the angle, θ, between these vectors is 30° as shown in Figure 5.24. Calculate **u** ▪ **v.**

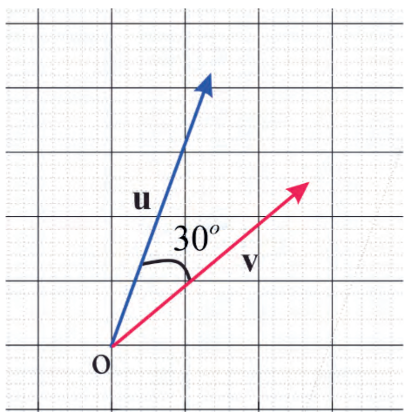


Figure 5.24

Solution

$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$

$= 8\times 7 \cos 30$

$= 56(\frac{\sqrt{3}}{2})$

$= 28\sqrt{3}.$

Example 2

Find the dot product of the vectors **u** and **v** when

a. **u** = (0, 2) and **v** = (0, 4)

b. **u** = (3, 0) and **v** = (2, $2\sqrt{3}$)

Solution

From the Theorem 5.4 you have

a. $|\mathbf{u}| = 2$, $|\mathbf{v}| = 4$ and $\boldsymbol{\theta} = 0^\circ \Rightarrow \mathbf{u} \cdot \mathbf{v} = 2 \times 4 \times \cos 0^\circ = 8 \times 1 = 8$.

b. Let us show these vectors graphically as shown in Figure 5.25.

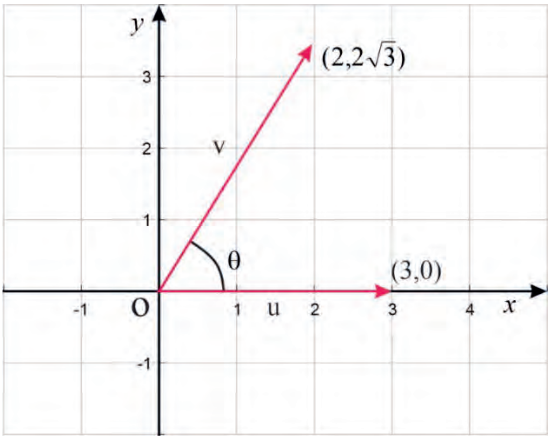


Figure 5.25

$|\mathbf{u}| = 3$, $|\mathbf{v}| = \sqrt{(2\sqrt{3})^2 + 2^2} = 4$ and $\cos\theta = \frac{2}{4} = \frac{1}{2}$ why?

Then, $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$

$= 3 \times 4 \times \frac{1}{2} = 6$

Before discussing the next properties of dot product, recall the relation between degree and radian that you have learnt in grade 10 such as, $45^\circ = \frac{\pi}{4}$, $90^\circ = \frac{\pi}{2}$, $180^\circ = \pi$, $270^\circ = \frac{3\pi}{2}$ and $360^\circ = 2\pi$.

Note

1. $\mathbf{i} \cdot \mathbf{j} = 0$, since $\cos\left(\frac{\pi}{2}\right) = 0$.
2. $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$, since $\cos(0) = 1$.
3. If either $\mathbf{u}$ or $\mathbf{v}$ is 0 then $\mathbf{u} \cdot \mathbf{v} = 0$.
4. $\mathbf{u} \cdot \mathbf{u} > 0$ if $\mathbf{u} \neq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if $\mathbf{u} = 0$.
5. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ …. dot product of vector is commutative.
6. If the vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel and pointing to the same direction, then $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|$.
7. If the vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel and pointing to the opposite direction, then $\mathbf{u} \cdot \mathbf{v} = -|\mathbf{u}||\mathbf{v}|$.
8. In particular, for any vector $\mathbf{u}$, you have $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$, you write $\mathbf{u}^2$ to mean $|\mathbf{u}|^2$
9. If the vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular or orthogonal, then $\mathbf{u} \cdot \mathbf{v} = 0$, since $\cos\left(\frac{\pi}{2}\right) = 0$.
10. If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, then the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$ is given by $\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$.

Exercise 5.8

1. Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors with $|\mathbf{u}| = 4$ and $|\mathbf{v}| = 2$ and $\theta = 45°$.
   Find their dot product.
2. Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors with $|\mathbf{u}| = 7$ and $|\mathbf{v}| = 3$ and $\theta = \frac{\pi}{2}$.
   Find their dot product.
3. Find the dot product of the vectors $\mathbf{u}$ and $\mathbf{v}$ when
   a. $\mathbf{u} = (2, 0)$ and $\mathbf{v} = (4, 0)$.
   b. $\mathbf{u} = (0, 3)$ and $\mathbf{v} = (2\sqrt{3}, 2)$.

Theorem 5.5 (Properties of the dot product)

If **u, v** and **w** are vectors and k is a scalar then

1. $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$
2. Distributive property

   i) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

   ii) $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u}$

Corollary 5.1

If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are vectors, then $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2$.

Proof

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j}) \cdot (v_1\mathbf{i} + v_2\mathbf{j}) \\ &= u_1\mathbf{i} \cdot (v_1\mathbf{i} + v_2\mathbf{j}) + u_2\mathbf{j} \cdot (v_1\mathbf{i} + v_2\mathbf{j}) \\ &= u_1\mathbf{i} \cdot v_1\mathbf{i} + u_1\mathbf{i} \cdot v_2\mathbf{j} + u_2\mathbf{j} \cdot v_1\mathbf{i} + u_2\mathbf{j} \cdot v_2\mathbf{j} \\ &= u_1v_1\mathbf{i} \cdot \mathbf{i} + u_1v_2\mathbf{i} \cdot \mathbf{j} + u_2v_1\mathbf{j} \cdot \mathbf{i} + u_2v_2\mathbf{j} \cdot \mathbf{j} \\ &= u_1v_1 + u_2v_2.\end{aligned}$$

Example 3

Find the dot product of the vectors $\mathbf{u} = 5\mathbf{i} + 3\mathbf{j}$ and $\mathbf{v} = 7\mathbf{i} - 2\mathbf{j}$.

Solution

$\mathbf{u} \cdot \mathbf{v} = (5\mathbf{i} + 3\mathbf{j}) \cdot (7\mathbf{i} - 2\mathbf{j}) = 5 \times 7 + 3 \times (-2) = 29.$

Example 4

Find the cosine of the angle formed by two vectors $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j}$.

Solution

From the corollary above you have

$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 = 2(-2) + 3(3) = 5$

The norm of vector **u** is $\sqrt{2^2 + 3^2} = \sqrt{13}$ and the norm of vector **v** is

$\sqrt{(-2)^2 + 3^2} = \sqrt{13}$.

Using the Theorem for scalar product, you have

$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$

Then, $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{5}{\sqrt{13} \cdot \sqrt{13}} = \frac{5}{13}$

Therefore, $\cos \theta = \frac{5}{13}$.

Example 5

Let **u** and **v** be two vectors $|\mathbf{u}| = 3$ and $|\mathbf{v}| = 6$ and $\theta = \frac{\pi}{3}$.

a. Find $|3\mathbf{u} - 2\mathbf{v}|$.

b. Find the cosine of angle between $3\mathbf{u} - 2\mathbf{v}$ and **u.**

Solution

a. Using the properties of dot product, you have

$$\begin{aligned} |3\mathbf{u} - 2\mathbf{v}|^2 &= (3\mathbf{u} - 2\mathbf{v}) \cdot (3\mathbf{u} - 2\mathbf{v}) \\ &= 9\mathbf{u}^2 - 12\mathbf{u} \cdot \mathbf{v} + 4\mathbf{v}^2 \\ &= 9 \cdot 3^2 - 12|\mathbf{u}||\mathbf{v}| \cos\frac{\pi}{3} + 4 \cdot 6^2, \text{ since } \mathbf{u}^2 = |\mathbf{u}|^2 \\ &= 81 - 12 \cdot 3 \cdot 6 \cdot \frac{1}{2} + 144 \\ &= 117 \end{aligned}$$

Thus, $|3\mathbf{u} - 2\mathbf{v}| = \sqrt{117}$.

b. Let θ be the angle between $3\mathbf{u} - 2\mathbf{v}$ and **u**. Then

$(3\mathbf{u} - 2\mathbf{v}) \cdot \mathbf{u} = |3\mathbf{u} - 2\mathbf{v}||\mathbf{u}| \cos \theta$

$\Rightarrow 3\mathbf{u}^2 - 2\,\mathbf{v} \cdot \mathbf{u} = |3\mathbf{u} - 2\mathbf{v}|\,(3) \cos \theta$ but from **a** you have $|3\mathbf{u} - 2\mathbf{v}| = \sqrt{117}$

$\Rightarrow 3 \cdot 3^2 - 2|\mathbf{v}||\mathbf{u}| \cos\frac{\pi}{3} = \sqrt{117} \cdot 3 \cos\theta$

$\Rightarrow 3 \cdot 3^2 - 2 \times 6 \times 3 \times \frac{1}{2} = 3\sqrt{117} \cos\theta$

$\Rightarrow 27 - 18 = 3\sqrt{117} \cos\theta$

$\Rightarrow 9 = 3\sqrt{117} \cos\theta$

$\Rightarrow \cos\theta = \frac{3}{\sqrt{117}} = \frac{3}{3\sqrt{13}} = \frac{1}{\sqrt{13}} = \frac{\sqrt{13}}{13}.$

Exercise 5.9

1. Find the dot product of the following vectors:

   a. $\mathbf{u} = 4\mathbf{i} + 2\mathbf{j}$ and $\mathbf{v} = 3\mathbf{i} - 5\mathbf{j}$ b. $\mathbf{u} = -2\mathbf{i} + 2\mathbf{j}$ and $\mathbf{v} = 7\mathbf{i} + 4\mathbf{j}$

2. Find the cosine of the angle formed by the following two vectors:

   a. $\mathbf{u} = 4\mathbf{i} + 3\mathbf{j}$ and $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$ b. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$

3. Let $\mathbf{u} = (1, -1)$, $\mathbf{v} = (1, 1)$ and $\mathbf{w} = (-2, 3)$. Find the cosines of the angles between:

   a. **u** and **v** b. **v** and **w** c. **u** and **w**

4. Say whether the vectors are orthogonal, parallel or neither

   a. $\mathbf{u} = (-5, 3)$ $\mathbf{v} = (6, -8)$ b. $\mathbf{u} = (4, 6)$ $\mathbf{v} = (-3, 2)$

   c. $\mathbf{u} = (-1, \frac{1}{2})$ $\mathbf{v} = (3, 6)$ d. $\mathbf{u} = (2, 6)$ $\mathbf{v} = (-3, -9)$

5. Find the angle between the two vectors $2\mathbf{i} + 3\mathbf{j}$ and $5\mathbf{i} - 2\mathbf{j}$.

6. Show that $(\mathbf{u} - \mathbf{v})(\mathbf{u} + \mathbf{v}) = \mathbf{u}^2 - \mathbf{v}^2$.

7. Show that $(\mathbf{u} \mp \mathbf{v})^2 = \mathbf{u}^2 \mp 2\,\mathbf{u} \cdot \mathbf{v} + \mathbf{v}^2$, where $\mathbf{u}^2 = \mathbf{u} \cdot \mathbf{u}$.

8. Vectors **u** and **v** make an angle $\theta = \frac{2\pi}{3}$ between them. If $|\mathbf{u}| = 5$ and $|\mathbf{v}| = 12$, then find

   a. $\mathbf{u} \cdot \mathbf{v}$ b. $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$ c. $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$ d. $|3\mathbf{u} + \mathbf{v}|$

9. Given a triangle with vertices A (1, 5), B (2, 2), and C (4, 6), find angle of BCA.

5.3.2 Vectors in Three-dimensional Space

Coordinate of a point in space

Activity 5.8

1. Represent each of the following points on the $xy-$ coordinate plane.

   a. A (1,1)   b. B (2, 2)   c. C (−4,4)   d. D (0,−5)   e. E(−2, −2)

2. Plot the following points, by naming the vertical axis on the plane z, and the horizontal axis y.

   a. P (2, 3)   b. Q(−1, 3)   c. (−2,−2)

From activity 5. 8, you observed that two numbers are necessary to locate a point in a plane. You know that any point in the plane can be represented as an ordered pair (a, b) of real numbers, where a is the x-coordinate and b is the y-coordinate. For this reason, a plane is called two-dimensional. A plane coordinate system is extended to three dimensional spaces as follows: Consider a fixed-point O in space and three lines that are mutually perpendicular at the point O. The point O is called the origin; the three lines are now called the x-axis, the y-axis and the z-axis. It is common to have the x- and the y-axes on a horizontal plane and the z-axis vertical or perpendicular to the plane containing the x- and the y-axes at the point O as shown in Figure 5.26 below.

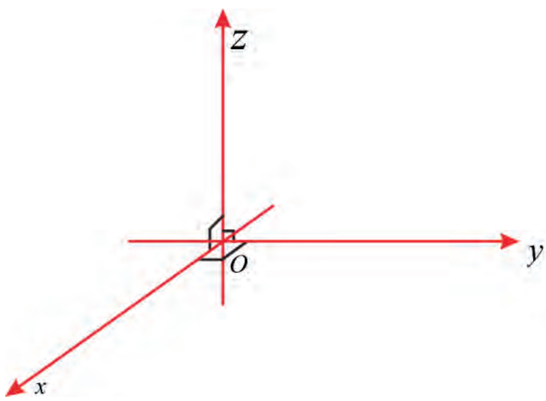


Figure 5.26

The direction of the z-axis is determined by the **right-hand rule** as illustrated in Figure 5.27:

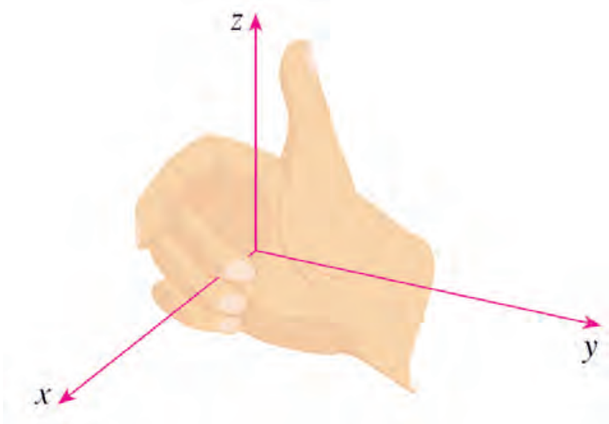


Figure 5.27

If you curl the fingers of your right hand around the z-axis in the direction of a 90° counterclockwise rotation from the positive x-axis to the positive y-axis, then your thumb points

in the positive direction of the z-axis. With these axes any point p in space can be assigned three coordinates p (x, y, z) as shown Figure 5.28.

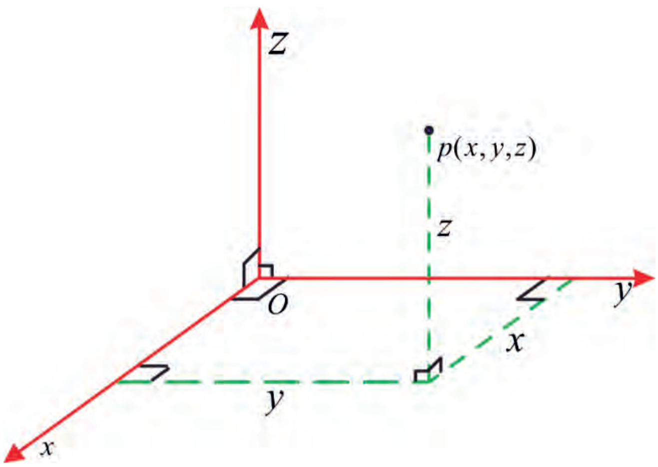


Figure 5.28

Example 1

Locate the point P (1, 3, 4) in space using the reference axes x, y and z.

Solution

The process of locating the point P may be described as follows:

The process of locating the point P may be described as follows: Start from the origin O and move 1 unit in the direction of the positive x-axis. Then move 3 units in the direction of the positive y-axis and finally move 4 units up in the direction of the positive z-axis to get point P (1, 3, 4) as shown in Figure 5.29.

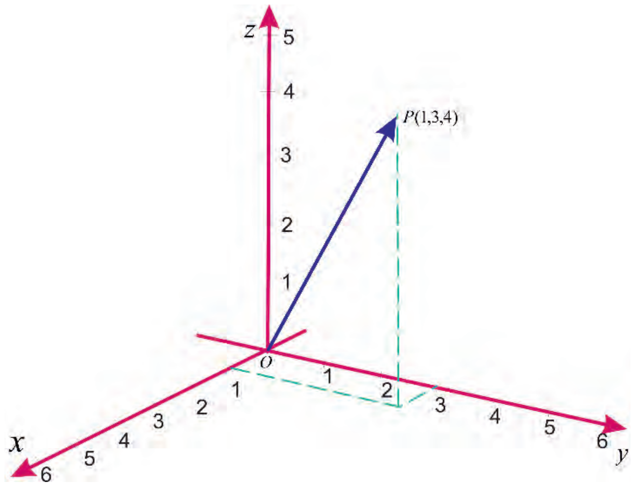


Figure 5.29

Exercise 5.10

Locate each of the following points in space using reference axes x, y and z. You may use the same or different coordinate systems in each case.

a. P (2, 1, 2)

b. Q (–3, 5, 4)

c. R (2, – 2, 3)

d. S ((2, $-\frac{1}{2}$,–4)

e. T (0, 0, –4)

f. Q (0, –4, 0)

Vectors in Space

Activity 5.9

1. How do you represent a vector on a plane?
2. How do you represent the magnitude of a vector?
3. How do you express the vector P (x, y) using the standard unit vectors **i** and **j**?

From activity 5.9, you have observed that the vector $\overrightarrow{OP}$ can be named using a single letter. That is, $\overrightarrow{OP}$ may be named P, or simply by P so that $\vec{P} = x\mathbf{i} + y\mathbf{j}$. Just as you worked with vectors on a plane by using the coordinates of their terminal points, you can handle vectors in a three-dimensional space with the help of the coordinates of the terminal points.

Now, let the initial point of a vector in space be the origin O of the coordinate system and let its terminal point be at P (x, y, z) as shown in Figure 5.30.

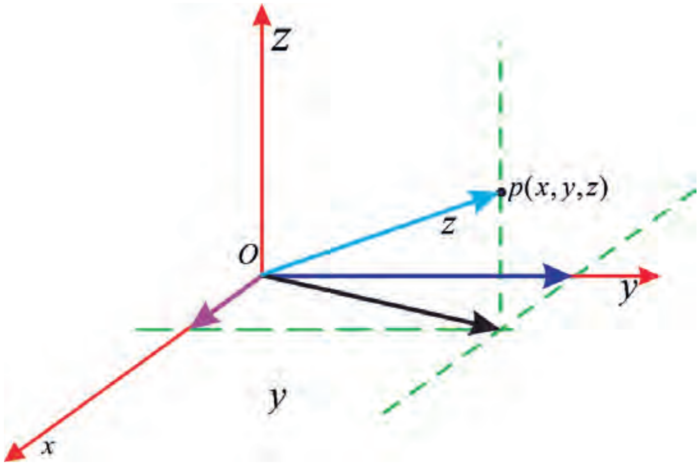


Figure 5.30

Then the vector $\overrightarrow{OP}$ can be expressed as the sum of its three components in the directions of the x, the y and the z-axis, in the form:

$\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$ are standard Unit vectors in the directions of the positive x, positive y and positive z-axis respectively as shown in Figure 5.31.

Do you observe that the vector $\overrightarrow{OP}$ is the sum of the three perpendicular vectors $\overrightarrow{OC}$, $\overrightarrow{OB}$ and $\overrightarrow{OD}$?

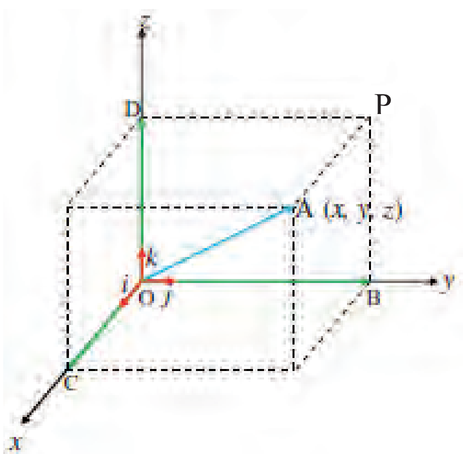


Figure 5.31

Example 2

If the initial point of a vector in space is at the origin and its terminal point or head is at P (4, 6, 5), show the vector using a coordinate system and identify its three perpendicular components in the directions of the three axes.

Solution

The three components are the vectors with common initial point O (0, 0, 0) and terminal points A (4, 0, 0) on the x-axis, B (0, 6, 0) on the y-axis and C (0, 0, 5) on the z-axis as shown in Figure 5.32.

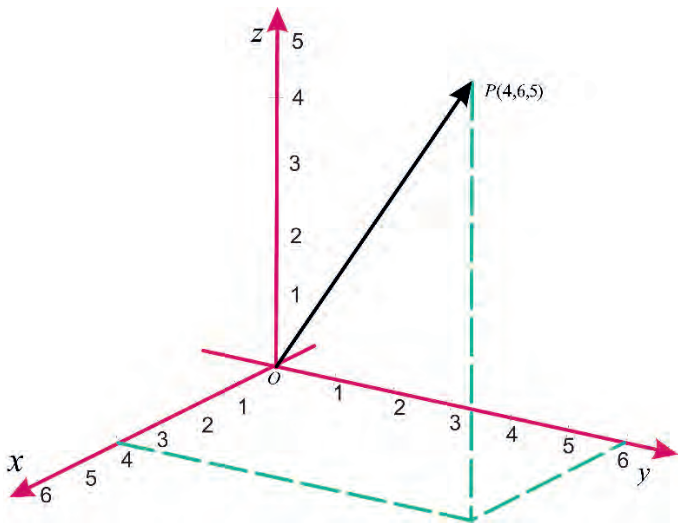


Figure 5.32

That means $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OD} + \overrightarrow{OP}$ or in terms of the unit vectors,

(4, 6, 5) = 4**i** + 6**j** + 5**k** = (4, 0, 0) + (0, 6, 0) + (0, 0, 5).

Magnitude of a vector

At the beginning of the discussion about vectors in space, it was mentioned that a vector is usually represented by an arrow, where the arrow head indicates the direction and the length of the arrow represents the magnitude of the vector. Thus, to find the magnitude of a vector, it will be sufficient to find the distance between the initial point and the terminal point of the vector in the coordinate space.

Example 3

If the initial point of a vector is at the origin of the coordinate space and the terminal point is at P (3, 2, 2), then find the magnitude of the vector $\overrightarrow{OP}$.

Solution

The magnitude of the vector $\overrightarrow{OP}$ is the distance from O to P. This is, $\sqrt{3^2 + 2^2 + 2^2} = \sqrt{17}$.

Note

1. If the initial point of a vector **u** is at the origin and its terminal point is at a point
   Q(x, y, z) or if $\mathbf{u} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then $|\mathbf{u}| = \sqrt{x^2 + y^2 + z^2}$.
2. If the initial point of **u** is at P(x_1, y_1, z_1) and the terminal point at Q(x_2, y_2, z_2) then $|\mathbf{u}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Exercise 5.11

1. Calculate the magnitude of each of the following vectors.
   a. (–1, 2, 0) b. (2, 1, –1) c. $(\frac{1}{2}, \frac{3}{4}, \frac{4}{5})$
2. Make a three-dimensional sketch showing each of the following vectors with initial point at the origin.
   a. $\mathbf{u} = 3\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ b. $\mathbf{v} = 4\mathbf{i} + 5\mathbf{j} - \mathbf{k}$
   c. $\mathbf{w} = -2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$ d. $\mathbf{u} = 3\mathbf{j} - 5\mathbf{k}$

5.3.3 Cross Product

Activity 5.10

1. What do you know about cross product of vectors?
2. Find det(A) if A $=\begin{bmatrix} 4 & -4 \\ -5 & 2 \end{bmatrix}$
3. Find det(A) if A $=\begin{bmatrix} 5 & -3 & 1 \\ 0 & 1 & 3 \\ -1 & 4 & -1 \end{bmatrix}$

From Activity 5.10, recall 2 by 2 and 3 by 3 matrix that you have already seen in previous unit.

The determinant of 2 by 2 matrix $\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}$ is defined by $u_1v_2 - u_2v_1$.

Again, a determinant of 3 by 3 matrix can be defined in terms of 2 by 2 determinants as follows:

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

Observe that each term on the right side of the above equation involves a number u_i in the first row of the determinant, and u_i is multiplied by 2 by 2 determinant obtained from the left side by deleting the row and column in which u_i appears. This implies that 2 by 2 determinants and the standard basis vectors

i = (1, 0, 0), **j** = (0, 1, 0) and **k** = (0, 0, 1) define the cross product of the vectors

$\mathbf{u} = u_1\mathbf{i} + u_2\,\mathbf{j} + u_3\,\boldsymbol{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\,\mathbf{j} + v_3\,\boldsymbol{k}$ as

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

$$= (u_2v_3 - u_3v_2)\,\mathbf{i} - (u_1v_3 - u_3v_1)\,\mathbf{j} + (u_1v_2 - u_2v_1)\,\mathbf{k}$$

Thus, the cross product of two vectors, say **u** × **v**, is equal to another vector at right angles to both vectors, and it happens in the three-dimensions.

Definition 5.7

Given two non-zero vectors, $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ then the cross product of **u** and **v** is the vector

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\,\mathbf{i} - (u_1v_3 - u_3v_1)\,\mathbf{j} + (u_1v_2 - u_2v_1)\,\mathbf{k}$$

Example 1

If $\mathbf{u} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}$, then find the cross product of $\mathbf{u} \times \mathbf{v}$.

Solution

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}\mathbf{k}$$

$$= (-15 - 28)\,\mathbf{i} - (-5 - 8)\,\mathbf{j} + (7 - 6)\mathbf{k} = -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}$$

Thus, the cross product of $\mathbf{u} \times \mathbf{v}$ is $-43\mathbf{i} + 13\mathbf{j} + \mathbf{k}$.

Example 2

Find $|\mathbf{u} \times \mathbf{v}|$ where $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and $\mathbf{v} = 7\mathbf{i} + 8\mathbf{j} + 9\mathbf{k}$.

Solution

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 5 \\ 7 & 8 & 9 \end{vmatrix} = ((4\times 9) - (5\times 8))\,\mathbf{i} - ((3\times 9) - (5\times 7))\,\mathbf{j} + ((3\times 8) - (4\times 7))\mathbf{k}$$

$$= (36 - 40)\,\mathbf{i} - (27 - 35)\,\mathbf{j} + (24 - 28)\,\mathrm{k} = -4\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$$

$$= -4\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$$

Therefore, $|\mathbf{u} \times \mathbf{v}| = \sqrt{(-4)^2 + 8^2 + (-4)^2} = 4\sqrt{6}$.

Theorem 5.6 (Properties of the cross product)

If **u, v** and **w** are vectors and r is a scalar

1. Anticommutativity: $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. Multiplication by scalars: $(r\mathbf{u}) \times \mathbf{v} = r(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (r\mathbf{v})$
3. Distributivity: $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
4. Orthogonal properties: $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$
5. Cross product of the zero vector: $\mathbf{u} \times 0 = 0$
6. Cross product of the vector with itself: $\mathbf{u} \times \mathbf{u} = 0$
7. The cross product of the unit vectors: $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$
8. Triple scalar product: $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

Note

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$, $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$ and $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$

Exercise 5.12

1. If $\mathbf{u} = \frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \frac{3}{4}\mathbf{j} - 2\mathbf{k}$, then find the following.

   a. $\mathbf{u} \times \mathbf{v}$ b. $-(\mathbf{v} \times \mathbf{u})$ c. $\mathbf{u} \times \mathbf{u}$

   d. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$ e. $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$ f. $|\mathbf{u} \times \mathbf{v}|$

2. Let $\mathbf{u} = \mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{v} = 5\mathbf{i} + 6\mathbf{j} - 6\mathbf{k}$ be vectors. Then, find $\mathbf{u} \times \mathbf{v}$ and $|\mathbf{u} \times \mathbf{v}|$.
3. Find $\mathbf{u} \times \mathbf{v}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.
4. Given a vector $\mathbf{u} = (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) \times (\mathbf{i} + 3\mathbf{k})$ then find the magnitude of $\mathbf{u}$.

Theorem 5.7

The cross-product $\mathbf{u} \times \mathbf{v}$ is perpendicular or orthogonal to both **u** and **v**.

If **u** and **v** are represented by directed line segments with the same initial point then the cross-product $\mathbf{u} \times \mathbf{v}$ points in a direction perpendicular to the plane through **u** and **v**.

Its magnitude determines area of the parallelogram $A = |\mathbf{u} \times \mathbf{v}|$ as shown in Figure 5.33.

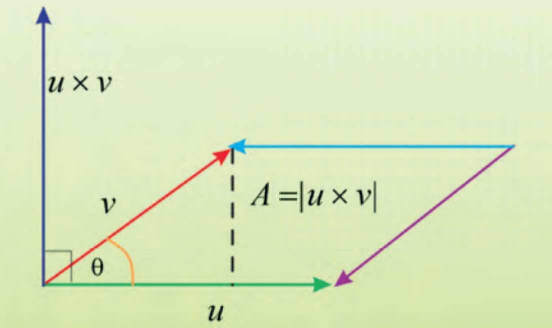


Figure 5.33

Example 3

Find the area of the parallelogram formed by the vectors $\mathbf{u} = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$, spanned between them.

Solution

Area of the parallelogram $= |\mathbf{u} \times \mathbf{v}|$

$$|\mathbf{A}| = |\mathbf{u} \times \mathbf{v}| = \left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 0 \\ 2 & 3 & 0 \end{vmatrix} \right|$$

$$= |(0)\,\mathbf{i} - (0)\,\mathbf{j} + (7)\,\mathbf{k}|$$

$$= 7 \text{ square units}$$

Example 4

If **u** and **v** are two vectors represented as $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, find the area of the parallelogram spanned between them.

Solution

Area of the parallelogram A = |**u** × **v** |

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 1 & 3 & 5 \end{vmatrix} = |(15 - 12)\,\mathbf{i} - (10 - 4)\,\mathbf{j} + (6 - 3)\,\mathbf{k} = 3\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$$

Then, $A = |\mathbf{u} \times \mathbf{v}| = |3\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}| = \sqrt{3^2 + (-6)^2 + 3^2}$

$= 3\sqrt{6}$.

Theorem 5.8

If θ is the angle between **u** and **v** (where, $0 \leq \theta \leq \pi$) then $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$.

Since a vector is completely determined by its magnitude and direction, we can now say that **u** × **v** is the vector that is perpendicular to both **u** and **v**, whose orientation is determined by the right-hand rule, and whose length is | **u** | | **v** |sinθ as shown in Figure 5.34.

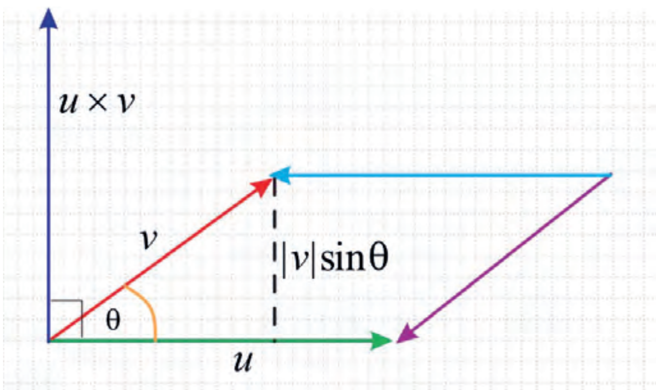


Figure 5.34

If **u** and **v** are represented by directed line segments with the same initial point, then they determine a parallelogram with base | **u** |, altitude | **v** |sin θ, and area

$A = |\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$ as shown in the Figure 5. 34.

Exercise 5.13

1. Find the area of the parallelogram defined by the vectors $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$.
2. Find the area of the triangle with vertices $(-1, 2)$, $(4, -1)$ and $(8, 7)$.
3. Find the area of the parallelogram defined by the vectors $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

5.4 Application of Scalar and Cross Product

The dot product is used to find the angle between two vectors, the length of the vector as well as work done by the force, whereas the cross product is mostly used to determine the vector, which is perpendicular to the plane surface spanned by two vectors.

Example 1

Find the measure of the angle between each pair of the vectors.

a. $\mathbf{u} = 4\mathbf{i} - \mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} + 3\mathbf{j}$

b. $\mathbf{u} = 2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{v} = -\frac{1}{2}\mathbf{i} + \frac{3}{4}\mathbf{j}$

Solution

a. To find the cosine of the angle formed by two vectors. substitute the components of the vector into equation $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$

But $\mathbf{u}\cdot\mathbf{v} = (4\mathbf{i} - \mathbf{j})\cdot(5\mathbf{i} + 3\mathbf{j}) = 4(5) + (-1)(3) = 20 - 3 = 17$, $|\mathbf{u}| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$

and $|\mathbf{v}| = \sqrt{5^2 + (3)^2} = \sqrt{34}$

$\Rightarrow \cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{17}{\sqrt{17}\sqrt{34}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$

$\Rightarrow \theta = 45°$.

b. To find the cosine of the angle formed by two vectors, substitute the components of the vector into equation $\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$

But $\mathbf{u} \cdot \mathbf{v} = (2\mathbf{i} + 5\mathbf{j})\,(-\frac{1}{2}\mathbf{i} + \frac{3}{4}\mathbf{j}) = 2(-\frac{1}{2}) + 5(\frac{3}{4}) = -1 + \frac{15}{4} = \frac{11}{4}$,

$|\mathbf{u}| = \sqrt{2^2 + 5^2} = \sqrt{29}$ and

$|\mathbf{v}| = \sqrt{(-\frac{1}{2})^2 + (\frac{3}{4})^2} = \frac{1}{4}\sqrt{13}$

$\Rightarrow \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{\frac{11}{4}}{\sqrt{29} \times \frac{1}{4}\sqrt{13}} = \frac{11}{\sqrt{377}}$

$\Rightarrow \theta = \cos^{-1}(\frac{11}{\sqrt{377}})$

Example 2

Suppose a child is pulling a wagon with a force having a magnitude of 8 N on the handle at an angle of 60° , as shown in Figure 5.35. If the child pulls the wagon 50 m, then find the work done by the force.

Solution

According to physics formula, work done is got by dot product of force and displacement. Hence, the work done by the force F, acting at an angle θ = 60° form the line of motion which is given by

$W = |F_x|\,S = (|F| \cos\theta)\,S = |F|\,S \cos\theta$, where S is the displacement of the wagon.

$= 8\text{N}\,(50\text{ m}) \cos 60° = 400(\frac{1}{2})\text{ N m} = 200\text{ J}$

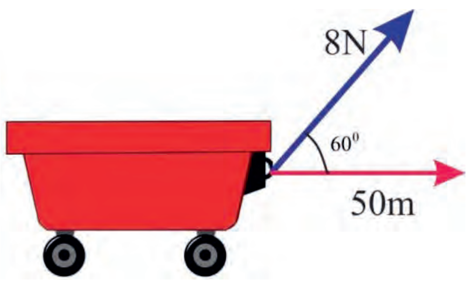


Figure 5.35

Example 3

Find the area of the triangle with vertices P (1, 4, 6), Q (–2, 5, –1), and R (1, –1, 1).

Solution

$\overrightarrow{PQ} = (-2-1)\,\mathbf{i} + (5-4)\,\mathbf{j} + (-1-6)\,\mathbf{k} = -3\mathbf{i} + \mathbf{j} - 7\mathbf{k}$

$\overrightarrow{PR} = (1-1)\,\mathbf{i} + (-1-4)\,\mathbf{j} + (1-6)\,\mathbf{k} = 0\mathbf{i} - 5\mathbf{j} - 5\mathbf{k}$

Then, $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = (-5-35)\,\mathbf{i} - (15-0)\,\mathbf{j} + (15-0)\,\mathbf{k}$

$= -40\mathbf{i} - 15\mathbf{j} + 15\mathbf{k}$

Thus, $\overrightarrow{PQ} \times \overrightarrow{PR} = -40\mathbf{i} - 15\mathbf{j} + 15\mathbf{k}$. The area of the parallelogram with adjacent sides $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is the magnitude of this cross product:

$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(-40)^2 + (-15)^2 + (15)^2}$

$= 5\sqrt{82}$

The area A of the triangle PQR is half the area of this parallelogram.
Thus, $A = \frac{5}{2}\sqrt{82}$,

Example 4

A bolt is tightened by applying a 40 N force to a 0.25 m wrench, as shown in Figure 5.36.

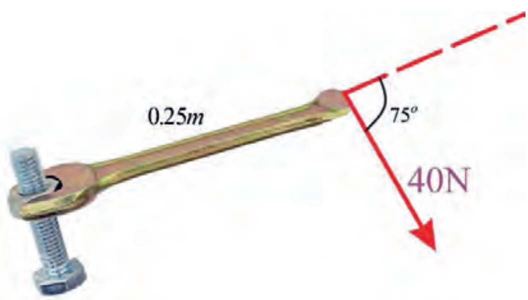


Figure 5.36

Find the magnitude of the torque about the centre of the bolt.

Solution

According to physics formula, the torque vector is calculated by cross product of the position vector and the force.

Thus, $|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin 75°$

$= (0.25)(40) \sin 75°$

$= 10 \sin 75° \approx 9.66$ N m,

If the bolt is right-threaded, then the torque vector itself is $\tau = |\tau|\,\mathbf{n} \approx 9.66\,\mathbf{n}$ where $\mathbf{n}$ is a unit vector directed down into the page.

Exercise 5.14

1. Find the measure of the angle between each pair of the vector.

   a. $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} - \mathbf{j}$ b. $\mathbf{u} = \mathbf{i} + 7\mathbf{j}$ and $\mathbf{v} = -\mathbf{i} + \mathbf{j}$

   c. $\mathbf{u} = 6\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{i} + 8\mathbf{j}$

2. A conveyor belt generates a force $\vec{F} = 5\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$ that moves a suitcase from point P (1, 1, 1) to Point Q (9, 4, 7) along a straight line. Find the work done by the conveyor belt. The distance is measured in meters and the force is measured in newtons.

3. If $\mathbf{u}$ and $\mathbf{v}$ are vectors represented as $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, then find the area of the parallelogram spanned between them.

4. Find the area of the triangle whose vertices are A (3, −1, 2), B (1, −1, −3) and C (4, −3, 1).

5. Find the area of the quadrilateral whose vertices are A (1, 0), B (3, 5), C (7, 2) and D (5, −2).

6. A car mechanic applies a force of 800 N to a wrench to loosen a bolt. She applies the force perpendicular to the arm of the wrench. The distance from the bolt to her hand is 0.40 m. What is the magnitude of the torque applied?

5.5 Application of Vectors

In previous section, you have seen that vectors have many applications. Geometrically, any two points in the plane determine a straight line. A straight line in the plane is determined by its slope and a point through which it passes. These lines have been determined to have a certain direction. Thus, related to vectors, you will see how one can write equations of lines, circles, and tangent line to a circle using vectors.

5.5.1 Vectors and Lines

Activity 5.11

Consider two points in the line such as $R_0\ (x_0\ , y_0)$ and $R_1\ (x_1\ , y_1)$, then the vector from R_0 to R_1 is $\mathbf{v} = (x_1 - x_0,\ y_1 - y_0)$ as shown in Figure 5.36. Discuss what will happen to the magnitude and direction of vector $\mathbf{v} = \vec{r_1} - \vec{r_0} = (x_1 - x_0,\ y_1 - y_0)$, where $\vec{r_1}$ and $\vec{r_0}$ are position vectors corresponding to the points R_0 and R_1 respectively, if t = 1, 2, 3, 4 and so on and generalize it for $\vec{r} = \vec{r_0} + t\mathbf{v}$ where t is a scalar, $t \in \mathbb{R}$ and **v** is a direction vector of the line.

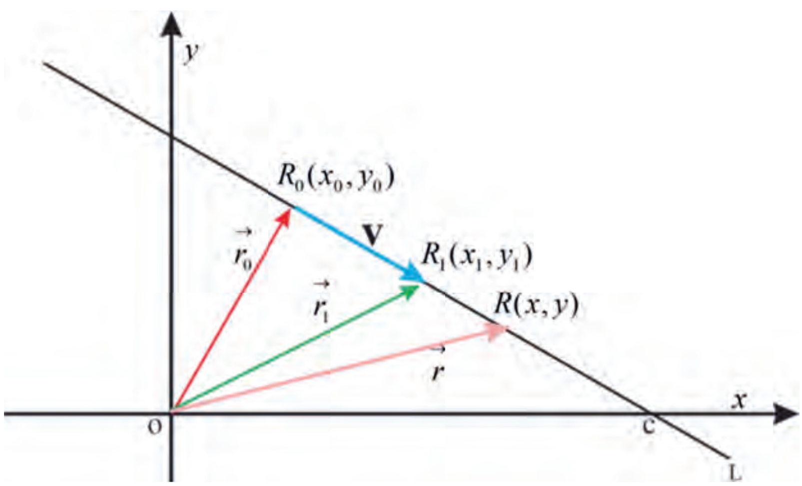


Figure 5.37

From Activity 5.11, you have observed that the line L that passes through R_0 and R_1 is parallel to the vector $\mathbf{v} = (x_1 - x_0, \ y_1 - y_0)$.

Let R (x, y) be any point on L. Then the position vector $(\vec{r})$ from the origin to any point on line L is obtained by

$\overrightarrow{OR_0} + \overrightarrow{R_0 R} = \overrightarrow{OR}$

$\Rightarrow \vec{r_0} + t\mathbf{v} = \vec{r}$

$\Rightarrow \vec{r} = \vec{r_0} + t\mathbf{v}$ where t is a scalar, $t \in \mathbb{R}$ and **v** is a direction vector of the line.

Thus,

1. If a direction vector **v** and a point $R_0(x_0, y_0)$ are given, then the vector equation of the line L determined by $\vec{r_0}$ and **v** is given by $\vec{r} = \vec{r_0} + t\mathbf{v}$, $t \in \mathbb{R}$ and $\mathbf{v} \neq \mathbf{0}$ where t is a scalar, $t \in \mathbb{R}$ and **v** is a direction vector of the line.
2. If $\mathbf{v} = (p, q)$, $R(x, y)$ and $R_0(x_0, y_0)$, then the above equation can be written as

   $(x, y) = (x_0, y_0) + t(p, q) = (x_0, y_0) + (tp, tq) = (x_0 + tp, y_0 + tq)$

   Then, from equality of vectors you have

   $x = x_0 + tp$ and $y = y_0 + tq$ where $t \in \mathbb{R}$, $(p, q) \neq (0,0)$

   This system of the equations is called the parametric equations of the line L through $R_0(x_0, y_0)$ whose direction is that of the vector $\mathbf{v} = (p, q)$, t is called a parameter.
3. If p and q are both different from 0 then from (2) above

   $$t = \frac{x - x_0}{p} \quad \text{and} \quad t = \frac{y - y_0}{q}$$

   $\Rightarrow \dfrac{x - x_0}{p} = \dfrac{y - y_0}{q}$ which is called the standard equations of the line.

Example 1

Consider the line passing through the point (1, 4) and (2, 2), then

a. Find the vector equation.

b. Find the parametric equations.

c. Find the standard equations.

Solution

Let $R_0 = (1, 4)$ and $R_1 = (2, 2)$ as shown in Figure 5.38.

a. The vector equation of the line is:

$$(x, y) = (1, 4) + t\big((2, 2) - (1, 4)\big)$$

$$= (1, 4) + t(1, -2)$$

Thus, $(x, y) = (1, 4) + t\big(1, -2\big), t \in \mathbb{R}$,

b. From the vector equation of question (a), you have the parametric equations:

$x = 1 + t, y = 4 - 2t, t \in \mathbb{R}$.

c. From parametric equation of question (b), you have the standard equation

$x - 1 = \frac{y-4}{-2} \Rightarrow 2x + y - 6 = 0$.

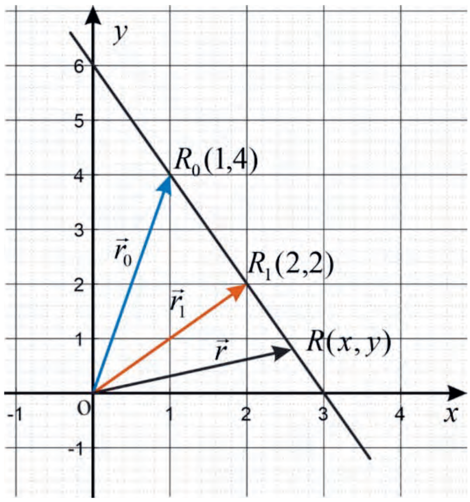


Figure 5.38

Example 2

Consider a line passing through (2, 2) and with direction vector (1, 4), then

a. Find the vector equation of the line.

b. Find the parametric equation.

Solution

Let $R_0 = (2, 2)$ and $\mathbf{v} = (1, 4)$ as shown in Figure 5.39.

a. The vector equation of the line is:

$(x, y) = (2, 2) + t\,(1, 4), t \in \mathbb{R}$.

b. The parametric equations are:

$x = 2 + t, y = 2 + 4t, t \in \mathbb{R}$.

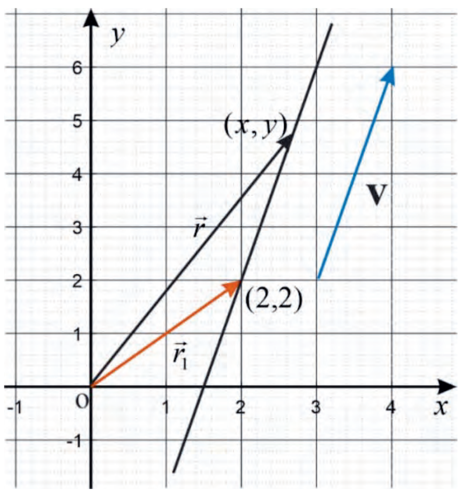


Figure 5.39

Exercise 5.15

1. Consider the line defined by the following points:

   a. (2, −9) and (8, 4)   b. (7, 3) and (4, −3)   c. (−1, −1) and (3, 10)

   1) Sketch the line.

   2) Find the vector equation of the line.

   3) Find the parametric equations of the line.

   4) Find the standard equation of the line.

2. Find the vector equation of the line passing through (1, −3) with direction vector (−5, 6).

3. Find the vector equation of line passing through the point having vector 5**i** + 4**j** and having direction vector (−3, 4).

4. Find the vector equation of the line passing through the point having position vector −**i** − **j** and parallel to the line L, **i** + 2**j**.

5.5.2 Vectors and Circles

Activity 5.12

Consider a circle with centre $C\ (x_0,\ y_0)$ and radius r> 0 and any point $R\ (x,\ y)$ on a circle such that $|\vec{r} - \vec{c}| = r$. where $\vec{r}$ and $\vec{c}$ are position vectors of $R\ (x,\ y)$ and $C\ (x_0,\ y_0)$ respectively as shown in Figure 5.40. Discuss how to show that $\vec{r}\cdot\vec{r} - 2\vec{r}\cdot\vec{c} + \vec{c}\cdot\vec{c} = r^2$ represents an equation of the circle at $C(x_0,\ y_0)$ and radius r through a position vector of any point on the circle $R(x,\ y)$.

Consider a circle with centre $C\ (x_0,\ y_0)$ and radius r > 0 as the set of points $R\ (x,\ y)$ in the plane such that $|\vec{r} - \vec{c}| = r$, where $\vec{r}$ and $\vec{c}$ are position vectors of $R\ (x,\ y)$ and $C\ (x_0,\ y_0)$, respectively as shown in Figure 5.40.

By squaring both sides $|\vec{r} - \vec{c}|^2 = r^2$

$\Rightarrow (\vec{r} - \vec{c}) \cdot (\vec{r} - \vec{c}) = r^2$ ……………………1

$\Rightarrow \vec{r} \cdot \vec{r} - 2\vec{r} \cdot \vec{c} + \; = r^2$ ……………………2

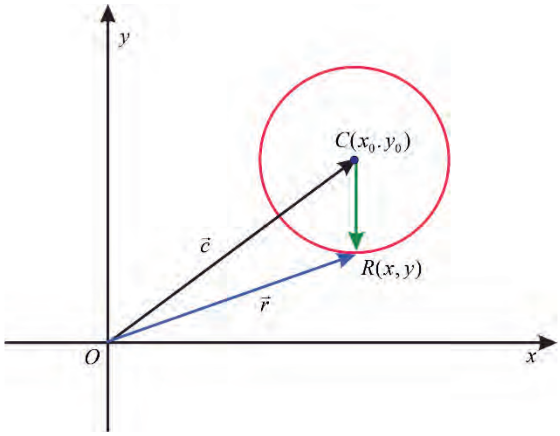


Figure 5.40

Thus, equation 2 represents the equation of the circle centred at $C\,(x_0, y_0)$ and radius r by using a position vector of any point on the circle $R\,(x, y)$.

Substituting the corresponding components of $\vec{r}$ and $\vec{c}$ in equation 1 and simplification, we obtain:

$(x - x_0)^2 + (y - y_0)^2 = r^2$ which is called **the standard equation of a circle.**

This equation can be expressed as:

$x^2 + y^2 + Ax + By + C = 0$ where $A = -2x_0$, $B = -2y_0$ and $C = x_0^2 + y_0^2 - r^2$……………………by expanding and rearranging the terms.

Example 1

Find an equation of the circle centred at $C\,(2, 3)$ and radius 4.

Solution

Let $R\,(x, y)$ be a point on the circle as shown in Figure 5.41.

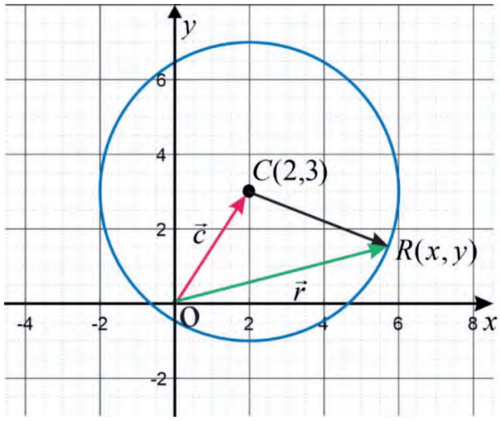


Figure 5.41

Let $\vec{r}$ and $\vec{c}$ be the position vectors of R (x, y) and C (x_0, y_0) respectively. Then from equation 2, you have

$\vec{r} \cdot \vec{r} - 2\vec{r} \cdot c + \vec{c} \cdot \vec{c} = r^2$

Substituting the corresponding components of $\vec{r}$ and $\vec{c}$ in equation 2, you obtain:

$(x, y) \cdot (x, y) - 2(x, y) \cdot (2, 3) + (2,3) \cdot (2,3) = 4^2$

$\Rightarrow x^2 + y^2 - 2(2x + 3y) + 4 + 9 = 16$ -------by corollary 5.1

$\Rightarrow x^2 + y^2 - 4x - 6y + 13 = 16$

$\Rightarrow x^2 + y^2 - 4x - 6y + 13 - 16 = 16 - 16$ ----------- subtracting both sides

$\Rightarrow x^2 + y^2 - 4x - 6y - 3 = 0.$

Example 2

Find the equation of the circle whose end points of a diameter are P $(-1, -1)$ and Q = $(4, 11)$.

Solution

The centre of the circle is $C\,(x_0, y_0) = C\left(\frac{-1+4}{2}, \frac{-1+11}{2}\right) = C\left(\frac{3}{2}, 5\right)$

The radius of the circle is given by $r = \frac{|d|}{2} = \frac{\sqrt{(4+1)^2 + (11+1)^2}}{2} = \frac{\sqrt{(5)^2 + (12)^2}}{2}$

$$= \frac{\sqrt{169}}{2} = \frac{13}{2}$$

Let $R(x, y)$ be a point on the circle and $\vec{r}$ and $\vec{c}$ be position vectors of R and C respectively.

Substituting the corresponding components of $\vec{r}$ and $\vec{c}$ in equation 2, you obtain:

$(x, y) \cdot (x, y) - 2(x, y) \cdot (\frac{3}{2}, 5) + (\frac{3}{2}, 5) \cdot (\frac{3}{2}, 5) = (\frac{13}{2})^2$

$\Rightarrow x^2 + y^2 - 2\,(\frac{3}{2}x + 5y) + \frac{9}{4} + 25 = \frac{169}{4}$, -------by corollary 5.1

$\Rightarrow x^2 + y^2 - 3x - 10y - 15 = 0.$

Exercise 5.16

1. Find an equation of the circle centred at C (1, −3) and radius 6.
2. Find the equation of the circle whose end points of a diameter are P (2, 1) and Q = (6, 3).
3. Find an equation of the circle centred at origin and passing through (3, 3).
4. Find an equation of the circle centred at C (−4, 2) and passing through (2, −5).

5.5.3 Equations of Tangent Lines to a Circle

Activity 5.13

Consider a circle with center R_0 (x_0, y_0) and a radius of r units as shown in the figure 5.41. Discuss how to show that the equation of the tangent line is $(x - x_o)(x_1 - x_o) + (y - y_o)(y_1 - y_o) = r^2$ where (x_1, y_1) is a point of tangency to the circle at R_1 and (x, y) is any point on line L.

Consider a circle with center C (x_0, y_0) and a radius of r units as shown in the Figure 5.42.

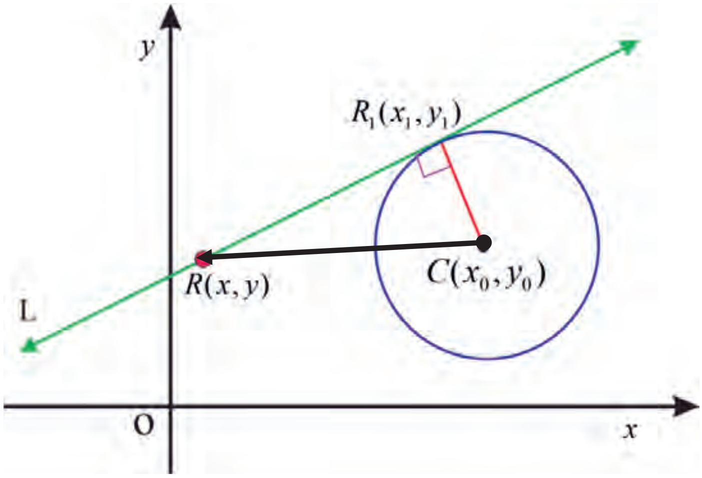


Figure 5.42

Let $R_1(x_1, y_1)$ be a point of tangency to the circle and it fulfills the following equation because it is on the point of the circle: $(x_1 - x_0)^2 + (y_1 - y_0)^2 = r^2$, $r > 0$.

A tangent line is a straight line that touches the circumference of a circle at the point of tangency.

The radius of the circle r is perpendicular (orthogonal) to the tangent L at the point of contact $R_1(x_1, y_1)$.

If $R\ (x, y)$ is an arbitrary point on L, $\overrightarrow{R_1R} \cdot \overrightarrow{CR_1} = 0$

Thus, the equation of the tangent line is

$$(x - x_1, y - y_1) \cdot (x_1 - x_0, y_1 - y_0) = 0$$

$\Rightarrow (x - x_1)(x_1 - x_0) + (y - y_1)(y_1 - y_0) = 0$ ………….. by corollary 5.1

Adding both sides of the equation above and $(x_1 - x_0)^2 + (y_1 - y_0)^2 = r^2$ you obtain

$$(x - x_1)(x_1 - x_o) + (y - y_1)(y_1 - y_o) + (x_1 - x_o)^2 + (y_1 - y_o)^2 = r^2$$

Factorizing $(x_1 - x_o)$ and $(y_1 - y_o)$ from respective terms, you have

$$(x - x_1 + x_1 - x_o)(x_1 - x_o) + (y - y_1 + y_1 - y_o)(y_1 - y_o) = r^2$$

$$\Rightarrow (x - x_o)(x_1 - x_o) + (y - y_o)(y_1 - y_o) = r^2.$$

Note

If the circle is centred at the origin, then the above equation becomes:

$xx_1 + yy_1 = r^2$.

Example 1

Find the equation of the tangent line to the circle $x^2 + y^2 + 10x + 2y + 13 = 0$ at the point $(-3, 2)$

Solution

Let $R_1(x_1, y_1) = R_1(-3, 2)$ as shown in Figure 5.43

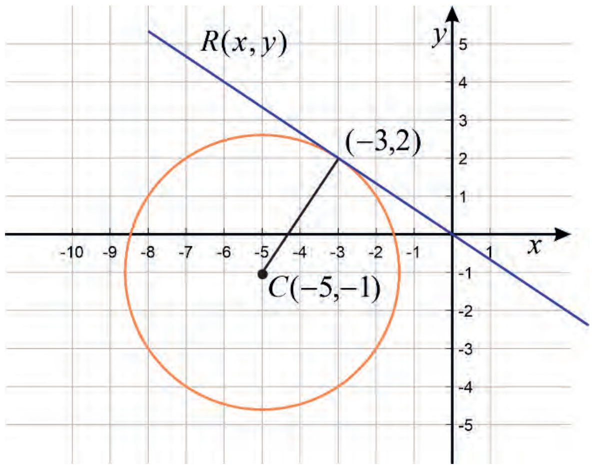


Figure 5.43

From $x^2 + y^2 + 10x + 2y + 13 = 0$

$\Rightarrow x^2 + 10x + y^2 + 2y + 13 = 0$

$\Rightarrow (x + 5)^2 - 25 + (y + 1)^2 - 1 + 13 = 0$ ……. By completing square

$\Rightarrow (x + 5)^2 + (y + 1)^2 - 26 + 13 = 0$

$\Rightarrow (x + 5)^2 + (y + 1)^2 = 13$

Therefore, the equation of the circle is

$(x + 5)^2 + (y + 1)^2 = 13$ with centre C $(-5, -1)$ and radius $r = \sqrt{13}$.

The equation of the tangent line is given by $(x - x_o)(x_1 - x_o) + (y - y_o)(y_1 - y_o) = r^2$

Substituting the corresponding components of $R_1(-3, 2)$

and R_0 $(-5, -1)$ in equation above, you obtain:

$$(x + 5)(-3 + 5) + (y + 1)(2 + 1) = \sqrt{13}^2$$

$$\Rightarrow (x + 5)(2) + (y + 1)(3) = 13$$

$$\Rightarrow 2x + 10 + 3y + 3 = 13$$

$$\Rightarrow 2x + 3y = 0$$

Thus, the tangent line to the graph of the circle at the point (−3, 2) is $2x + 3y = 0$

Example 2

Find equation of tangent line through R_1 (3, 4) a point on the circle $x^2 + y^2 = 25$ as shown in Figure 5.44

Solution

Recall that the equation of tangent line if the circle is centred at the origin is given by $xx_1 + yy_1 = r^2$ Since, the circle is centred at the origin with radius 5, the equation of the tangent line to the circle is

$x \times 3 + y \times 4 = 5^2 \Rightarrow 3x + 4y = 25.$

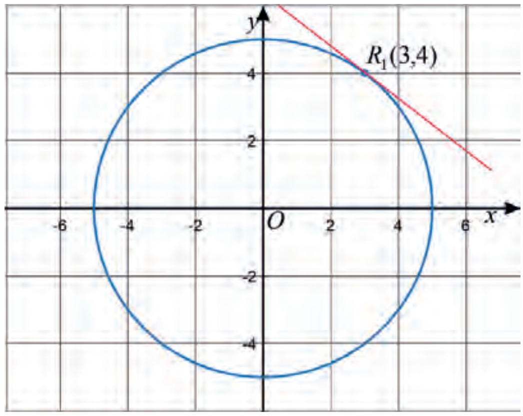


Figure 5.44

Exercise 5.17

1. Find the equation of the tangent line to the circle $x^2 + y^2 + 4x - 8y = 0$ at the point $R_1(-4, 8)$.
2. Find the equation of the line tangent to the circle $x^2 + y^2 - 8x - 10y - 128 = 0$ at the point $R_1(-8, 10)$.
3. Find equation of the tangent line through $R_1(-4, -3)$ point on the circle $x^2 + y^2 = 25$.
4. Find equation of tangent line to the circle $x^2 + y^2 = 29$ at the point R (−2, 5).

5.6 Applications

In this unit, you discussed definition and properties of scalar as well as cross product of vectors with their application, application of vectors such as determining the equations of a line, and the equations of a tangent line to a circle. Now, you will consider further applications involving vectors.

Example 1

Prove the diagonals of a parallelogram meet at right angles if and only if it is a rhombus.

Proof

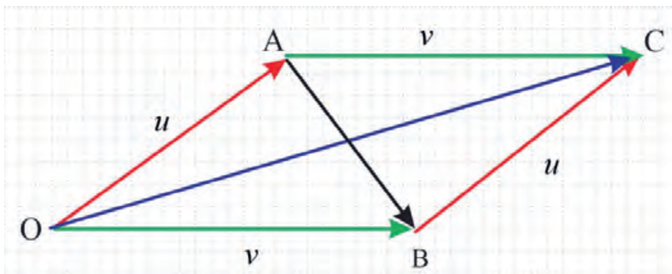


Figure 5.45

Let $\overrightarrow{OA} = \mathbf{u}$ and $\overrightarrow{OB} = \mathbf{v}$, as shown in Figure 5.45.

Therefore, $\overrightarrow{BC} = \overrightarrow{OA} = \mathbf{u}$

$$\overrightarrow{AC} = \overrightarrow{OB} = \mathbf{v}$$

$$\overrightarrow{OC} = \mathbf{u} + \mathbf{v}$$

$$\overrightarrow{AB} = \mathbf{v} - \mathbf{u}$$

$$\overrightarrow{OC} \cdot \overrightarrow{AB} = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{v} - \mathbf{u})$$

$$= \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} \text{ ----- distributive property}$$

$$= \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{u} = |\mathbf{v}|^2 - |\mathbf{u}|^2$$

Now for vectors to be perpendicular, the scalar product must be 0.

Therefore, $0 = |\mathbf{v}|^2 - |\mathbf{u}|^2$

$$\Rightarrow |\mathbf{v}|^2 = |\mathbf{u}|^2 \Rightarrow |\mathbf{v}| = |\mathbf{u}|$$

Hence, when the length of **u** is the same as **v** (i.e., a rhombus) the angle is 90°.

Example 2

A ball is thrown with an initial velocity of 70 m/s at an angle of 35° horizontally. Find the vertical and horizontal components of the velocity.

Solution

Let **v** represent the velocity and use the given information to write **v** in unit vector form:

$\mathbf{v} = 70\cos35°\boldsymbol{i} + 70\sin35°\boldsymbol{j}$

Simplify the scalars, you get:

$\mathbf{v} \approx 57.34\boldsymbol{i} + 40.15\boldsymbol{j}$ since the scalars are the horizontal and vertical components of **v**.

Therefore, the horizontal component is 57.34 m/s and the vertical component is 40.15 m/s.

Exercise 5.18

1. Prove the diagonals of a rectangle meet at right angles if and only if it is a square.
2. Use vectors to prove the sum of the squares of the lengths of a parallelogram's diagonals is equal to the sum of the squares of the lengths of the sides.
3. Ahmed is pulling her toy duck (mass 1kg) at a constant speed of 2.5 m/s. The string she uses to pull the duck makes an angle 45° above the horizontal and Ahmed keeps a constant tension in the string of 2 N. What is the work done by the tension force when the duck is pulled forward a distance of 2.8 m?
4. An airplane is flying in the direction of 43° east of north (also abbreviated as N43E at a speed of 550 m/h. A wind with speed 25 m/h comes from the southwest at a bearing of N15E. What is the ground speed and new direction of the airplane?

Problem Solving

1. A plane leaves the airport on the bearing of 45° traveling at 400 km/h if the bearing angles are measured clockwise from north. The wind is blowing at a bearing of 135° at a speed of 40 km/h. What is the actual velocity of the plane?
2. Two children are playing with a ball. The girl throws the ball to the boy. The ball travels in the air, curves 3 m to the right, and falls 5 m away from the girl, as shown in Figure 5.46. If the plane that contains the trajectory of the ball is perpendicular to the ground, find its equation.

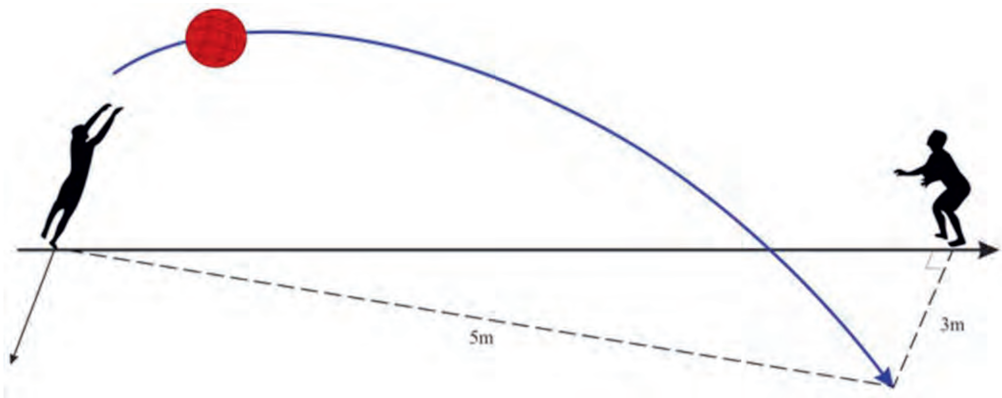


Figure 5.46

3. In square DEFG, side DE has column vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Find two possible column vectors for $\overrightarrow{EF}$.

Summary

1. **Vector**

   a. A scalar quantity is a physical quantity which can be expressed completely by its magnitude and particular units alone.

   b. A vector quantity is a physical quantity which can be expressed completely by stating both its magnitude with particular unit and directions.

   c. Two vectors are said to be equal, if they have the same magnitude and direction.

   d. A zero vector or null vector is a vector whose magnitude is zero and whose direction is indeterminate.

   e. A unit vector is a vector whose magnitude is one.

2. **Addition and subtraction of vectors**

   Let **u** and **v** be vectors, then the sum **u** + **v** is a vector given by the parallelogram law or triangle law satisfying the following properties.

   a. Vector addition is commutative. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

   b. Vector addition is associative. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

   c. $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
   d. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

   e. For any three vectors **u**, **v** and **w**, if $\mathbf{u} = \mathbf{v}$ and $\mathbf{v} = \mathbf{w}$, then $\mathbf{u} = \mathbf{w}$.

3. **Multiplication of a vector by a scalar**

   Let **u** and **v** be vectors, let k_1 and k_2 be scalars hen the following properties are true.

   a. $|k_1\mathbf{u}| = |k_1|\mathbf{u}$.
   b. Associative Property: $k_1(k_2\,\mathbf{u}) = (k_1k_2)\,\mathbf{u}$.

   c. Distributive Property: $(k_1 + k_2)\,\mathbf{u} = k_1\mathbf{u} + k_2\mathbf{u}$ or $k_1(\mathbf{u} + \mathbf{v}) = k_1\mathbf{u} + k_2\mathbf{v}$.

   d. Identity Property: $1 \cdot \mathbf{u} = \mathbf{u}$.
   e. Multiplicative Property of zero:$0 \cdot \mathbf{u} = 0$.

4. **Scalar product or dot product**

   If **u, v** and **w** be vectors, k is a scalar and θ is an angle between two vectors then the dot product of two vectors is defined as $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$ satisfying the following properties.

a. The scalar product of vectors is commutative. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

b. $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$.

c. The scalar product of vectors is distributive: $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

d. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$.

e. Two vectors **u** and **v** are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

f. If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are vectors, then $\mathbf{u}.\ \mathbf{v} = u_1v_1 + u_2v_2$.

5. **Cross product**

If **u, v** and **w** are vectors and r is a scalar the cross product of vectors has the following properties

a. Anticommutativity: $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

b. Multiplication by scalars: $(r\mathbf{u}) \times \mathbf{v} = r(\mathrm{u} \times \mathbf{v}) = \mathbf{u} \times (r\,\mathbf{v})$.

c. Distributivity: $\mathbf{u} \times (\mathbf{u} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.

d. Orthogonal properties: $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$.

e. Cross product of the zero vector: $\mathbf{u} \times 0 = 0$.

f. Cross product of the vector with itself: $\mathbf{u} \times \mathbf{u} = 0$.

g. The cross product of the unit vectors: $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$.

h. Triple scalar product: $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}).\ \mathbf{w}$.

Review Exercise

1. If vector **u** = (3, −2), **v** = (−2, −1) and **w** = (5, 1), then find

   a. 2**u** − 3**v**   b. |3**u** −2**v** + **w**| c. **u** + 4**v** −**w**   d. |2**u** + 2**w**|

   e. Find unit vector in the direction of **u** + **w**   f. Find **a** if **a** + **v** = 2**u** − **w**

2. If **u** = **i** + 2**j**, v = 2**i** + 4**j** and w = 5**i** – **j**, then find each of the following product

   a. (**u** ⋅ v) **w**   b. **u** ⋅ (3**w**)   c. $|\mathbf{v}|^2$

3. If **u** = 3**i** + 2**j** + 5**k** and **v** = **i** + 3**j** + 6**k**, then find

   a. **u** × **v**   b. **v** × **u**

4. If vector **v** = 2**i** + 3**j**, then find the angle formed by the following vectors:

   a. **v** and **i**   b. **v** and **j**

5. Determine whether **u** = **i** + 5**k** and **v** = 10**i** + 3**j** − 2**k** are orthogonal vectors or not.

6. Let **u** = (−5, 3, 7) and **v** = (6, −8, 2). Determine if the vectors are parallel, perpendicular or neither.

7. Given the vectors **u** = 6**i** + **j** − 3**k** and **v** = 2**i** + 3**j** + **k**, then find

   a. the area of the parallelogram formed by **u** and **v.**

   b. the area of the triangle formed by **u** and **v.**

8. A constant force of 60N is applied at angle of 30° to pull a handcart of 10 m across the ground, as shown in Figure 5.47. What is the work done by this force?

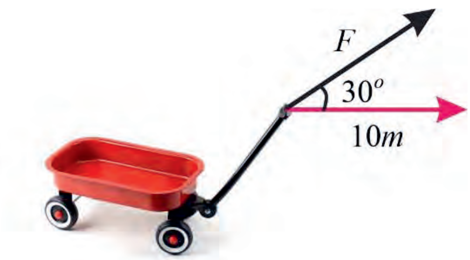


Figure 5.47

9. If the line L passing through the point P (−2, 1) and parallel to the vector 2**i** + 3**j**, then find   a. vector equation of the line.

   b. parametric equation of the line.

10. Find the equation of the line tangent to the circle $x^2 + y^2 + x - y = 7$ at point

    a. A (2 −5)   b. B (−1, 4)

11. If three forces $F_1 = 3\mathbf{i} - 4\mathbf{j}$, $F_2 = 4\mathbf{i} + 5\mathbf{j}$ and $F_3 = 5\mathbf{j}$ are acted on the body, then find the resultant force.

12. A balloon is rising vertically at 4 m/s. Wind is blowing from east to west at 3 m/s. Find the resultant velocity of the balloon.

13. A 50-N weight is hung by a cable so that the two portions of the cable make angles of 40° and 53°, respectively, with the horizontal, as shown in Figure 5.48. Find the magnitudes of the forces of tension T_1 and T_2 in the cables if the resultant force acting on the object is zero.

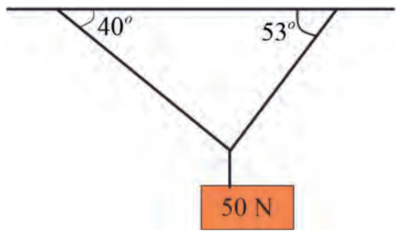


Figure 5.48

UNIT 6

TRANSFORMATIONS OF THE PLANE

Unit Outcomes

By the end of this unit, you will be able to

- Know basic concepts about transforming of the plane.
- Apply methods and procedures of transformation to transform plane figures.

Unit Contents

KEY Words

- identity transformation
- initial point
- non-rigid motion
- reflection
- rigid motion
- rotation
- standard position
- terminal point
- transformation
- vector translation

6.1 Introduction

In unit 5, you have discussed that various geometric and algebraic aspects of vector representation and vector algebra. In this unit, you focus on transformations and rigid motions which will be helpful to discuss translations, reflections and rotations.

Activity 6.1

Consider the following conditions and discuss what will happen to the shape or size or both of the object:

a. when a spring is compressed or stretched.

b. when the earth rotates about its axis.

c. when you see your image in a plane mirror.

d. when you draw your home's door.

From Activity 6.1, you have observed that some mappings of objects preserve shape, size or distance between any two points and some other mapping of objects are not preserve shape, size or distance between any two points. Due to this reason, transformations are classified as either rigid motion or non-rigid motion. But in this section, you will focus only on rigid motions.

Definition 6.1

A rigid motion is a motion which preserves distance.

That means, for A and B, AB = A'B' where A' and B' are the images of A and B respectively as shown in Figure 6.1

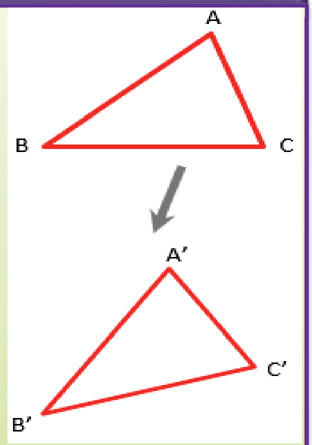


Figure 6.1

Note

Rigid motion carries any plane figures to a congruent plane figure. That means rigid motion carries triangles to congruent triangles, parallelograms to congruent parallelograms, etc.

Within the rigid motion, there are three main types of transformations that you will learn in this unit. These are translation, reflection and rotations.

6.2 Translation

Activity 6.2

Consider a triangle PQR and discuss what will happen to the shape, size and orientation of a triangle when you slide it as shown in Figure 6.2.

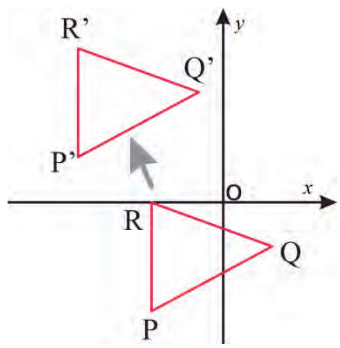


Figure 6.2

From Activity 6.2, you have observed that Δ PQR and $\Delta\ P'Q'R'$ have the same size, shape and orientation.

Definition 6.2

A translation is a transformation that occurs when every point of a figure is moved from one location to another location along the same direction through the same distance.

If point P is translated to point P', then the vector $\overrightarrow{PP'}$ is called **the translation vector**.

You can state a translation formula in terms of coordinates as follows:

Let T = $(\boldsymbol{a}, \boldsymbol{b})$ be a translation vector, then

a. The origin is translated to $(\boldsymbol{a}, \boldsymbol{b})$ That means $(0, 0) \xrightarrow{T(a,b)} (\boldsymbol{a}, \boldsymbol{b})$

b. The image of the point $P(x, y)$ under the translation vector T will be the point $P'(x + \boldsymbol{a}, y + \boldsymbol{b})$.

Example 1

1. Let T = (2, 3) be the translation vector. Find the image of the points A (2, 1), B (−1, 2) and C (−5, 1) through translation vector T.
2. If translation T takes the origin to (−2, 2) then find the image of the points P (3, 5) and Q (−1, 4).
3. The image of point (1, 2) under translation T is (2, 4). What is the image of the point (-2, 3) under the same translation?

Solution

1. To translate A (2, 1) through translation vector T = (2, 3), it has to move 2 units right and then 3 units up which is the point (4, 4) as shown in the Figure 6.3.

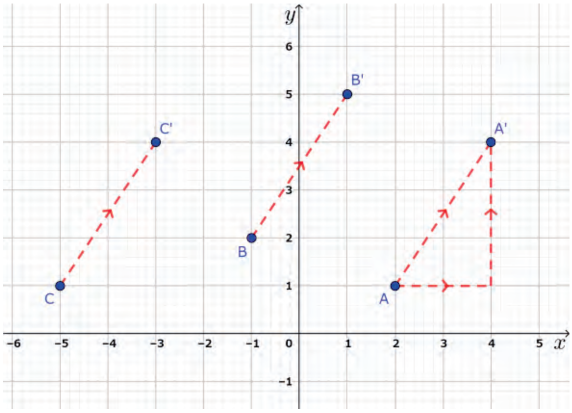


Figure 6.3

Hence, A' (4, 4) is the image of the point A (2, 1) under the translation vector T, $\overrightarrow{AA'} = (2, 3)$;

i.e., A (2,1) $\xrightarrow{T(a,\ b)}$ A' (4, 4)

Let us see the following table for some other points on the same graph and their corresponding images under the translation through translation vector T = (2, 3).

No.	Points in the Plane	Corresponding images
1	A (2, 1)	A' (4, 4)
2	B (-1, 2)	B' (1, 5)
3	C (-5, 1)	C' (-3, 4)
4	R (x, y)	R' $(x + 2, y + 3)$

From the above table, you can see that the image of any point under the translation T through translation vector (2, 3) is obtained by adding 2 to x-coordinate and 3 to y-coordinate of the given point; i.e., the image of R (x, y) is $R' =$ (x +2, y + 3)

Similarly, if the translation vector is $\overrightarrow{PQ}$ where P = (a, b) and Q = (c, d) then

a. the origin is translated to (c − a, d − b)

b. the point R (x, y) is translated to (x + c − a, y + d − b)

2. The image of point P (3, 5) is P’: $(3 + (-2), 5 + 2)$= (1, 7).

The image of point Q (−1, 4) is Q’: $((-1) + (-2), 4 + 2)$= (−3, 6).

3. The translation vector is $T = (2-1,\ 4-2) = (1,\ 2)$.

The image of point $(-2,\ 3)$ is $(-2+1,\ 3+2) = (-1,\ 5)$

Example 2

If A $(-4, -4)$, B $(-2, -1)$ and C $(-1, -5)$ are the vertices of a triangle ABC, find the coordinates of the image of ΔABC under the translation T = (6, 5). Draw ΔABC and its image on the same plane.

Solution

As A $(-4, -4)$, B $(-2, -1)$ and C $(-1, -5)$ are the vertices of ΔABC,

the coordinates of the vertices of image of ΔABC can be obtained by using the formula below:

you have, $P(x, y)=(x + a, y + b) =(x + 6, y + 5)$.

Hence, A': $(-4 + 6, -4 + 5) = (2, 1)$.

B': $(-2 + 6, -1 + 5) = (4, 4)$.

C': $(-1 + 6, -5 + 5) = (5, 0)$.

Graphically, you can show these points as shown in Figure 6.4.

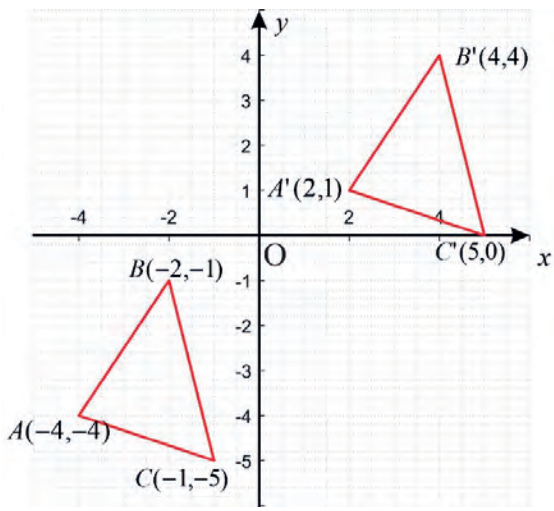


Figure 6.4

Exercise 6.1

1. Let $T = (2, 3)$ be the translation vector. Find the image of the pint A (−1, 4) and (3, 5) through the translation vector T.
2. The image of point (−2, 3) under translation T is (3, −1). What is the image of the point (4, 2) under the same translation?
3. A (−4, 2), B (0, 1) and C (−2, −3) are the vertices of ΔABC. If A' (−1, 6) be the image under the translation of vertex A, find the images B' and C' of B and C under the same translation.
4. Draw a square S (1, 2), Q (4, 1), R (5, 4) and E (2, 5). Find the image after the translation $(x, y) \rightarrow (x - 2, y + 3)$. Then, draw and label the image.
5. If a translation T takes the origin to (3, 4), then find the image of the points P (5, 7) and Q (−3, 6).

Line Translation

Example 3

If a translation T takes the point (−2, 3) to the point (1, 2), then what are the images of the following line

a. L: $3x - y + 4 = 0$

b. L: $4y + 2x + 1 = 0$

Solution

Let T be the translation vector, A = $(-2, 3)$ and $B = (1, 2)$. Then,

$T = \overrightarrow{AB} = B - A = (1-(-2), 2-3) = (3, -1)$. Thus, T:(3, − 1).

The point P (x, y) is translated to the point $P' = (x', y')$.

Here, $x' = x + 3$, $y' = y - 1$.

Therefore, $x = x' - 3$, $y = y' + 1$.

A translation map lines onto parallel lines. Let L' be the image of L under T. Then,

a. $L: 3x - y + 4 = 0$

$\Rightarrow \quad 3(x' - 3) - (y' + 1) + 4 = 0$

$\Rightarrow \quad 3x' - 9 - y' - 1 + 4 = 0$

$\Rightarrow \quad 3x' - y' - 6 = 0$

$\Rightarrow L': 3x - y - 6 = 0.$

b. $L: 4y + 2x + 1 = 0$

$\Rightarrow \quad 4(y' + 1) + 2(x' - 3) + 1 = 0$

$\Rightarrow \quad 4y' + 4 + 2x' - 6 + 1 = 0$

$\Rightarrow \quad 4y' + 2x' - 1 = 0$

$\Rightarrow L': 4y + 2x - 1 = 0.$

Note

Generally, a translation vector T: (a, b) translates point P (x, y) to point P$'(x', y')$.

The equation of P$'(x', y')$ can be got by P $\Rightarrow$ P$'$, $x' = x - a$, $y' = y - b$.

Exercise 6.2

1. If a translation T takes (1, –4) to (–3, 2), then find the image of the line L: $5x + 4y + 9 = 0$.
2. If a translation T takes the origin to (0, 2), then find the image of each of the following lines.

   a. $y = 3x - 1$ b. $3y + x = 12$
3. If the image of the line $2x - 3y = 7$ under a translation is $2x - 3y = 0$, then what is the translation vector of the translation line which is parallel to the x − axis or y −axis?

Circle Translation

Example 4

If a translation T takes the origin to (−2, 2), then find the equation of the image for the circle whose equation is $x^2 + y^2 = 4$.

Solution

Let T be the translation vector. The image of (x, y) under T is T $(x, y) = (x - 2, y + 2)$ as shown in Figure 6.5.

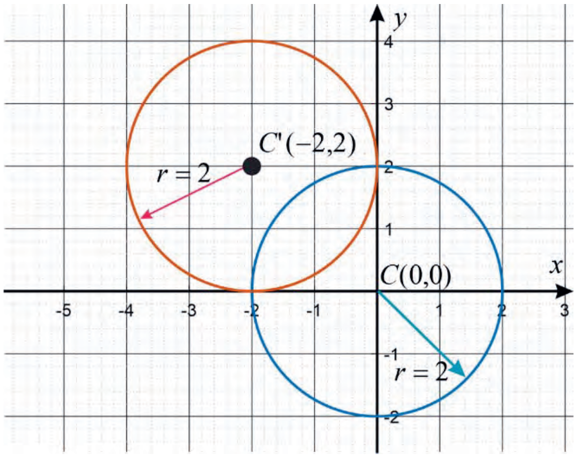


Figure 6.5

The centre of the circle (0, 0) is translated to (−2, 2)

Therefore, the image of the circle given by $x^2 + y^2 = 4$ is $(x + 2)^2 + (y - 2)^2 = 4$.

Note

As a translation is a rigid motion, to translate a circle it suffices to translate the center of the circle by the translation vector T (a, b) and keep its radius for the image too.

Exercise 6.3

1. If a translation T takes the origin to $(-3, 3)$, then find the equation of the image for the circle whose equation is $x^2 + y^2 = 16$.
2. If a translation T takes the origin to $(-6, 8)$, then find the equation of the image for the circle whose equation is $x^2 + y^2 = 9$.
3. Find the equation of the image of the circle $(x + 4)^2 + (y + 2)^2 = 7$ when translated by the vector $\overrightarrow{AB}$ where A = (2, 3) and B = (–5, 4).
4. Translate a circle $x^2 + y^2 - 4y = 21$ by the translation vector T $(3, -5)$. Find the image circle.

6.3 Reflection

Activity 6.3

Consider a triangle ABC and discuss what will happen to the shape, size and orientation of a triangle when you fold or flip over the fixed line as shown in Figure 6.6.

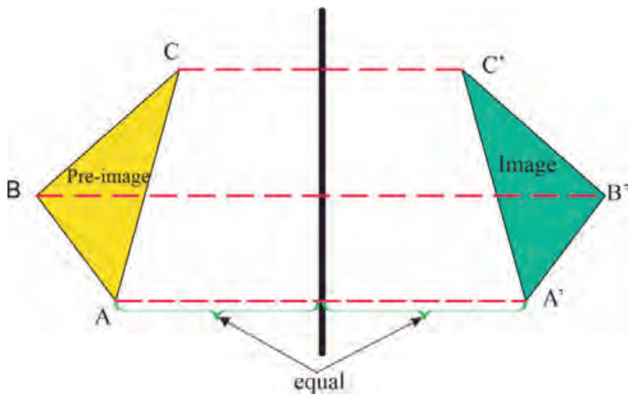


Figure 6.6

From Activity 6.3, you have observed that Δ ABC and its reflection (Δ A'B'C') have the same shape and size, but the figures face in opposite directions. Having these concepts, let us define reflection as follows.

Definition 6.3

Let L be a fixed line in the plane. A reflection **A** about a line **L** is a transformation of the plane onto itself which carries point **A** of the plane into the point **A'** of the plane such that L is the perpendicular bisector of $\overline{AA'}$.

The line L is called the **line of reflection or the axis of reflection**

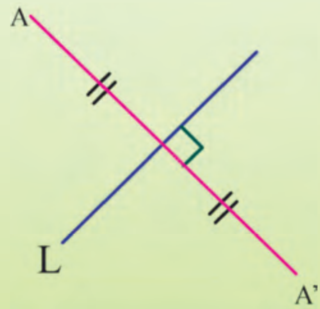


Figure 6.7

Notation: The reflection of a point **A** about the line **L**, is denoted by M(**A**);
i.e., A' = M(**A**).

Basic Properties of Reflection

1. A reflection about a line L has the property that, for two points **A** and **B** in the plane, if **A** = **B**, then M(**A**) = M(**B**). Hence, **reflection** is a function mapping the set of points in the plane into the set of points in the plane except reflection of points about the axis of reflection.
2. A reflection about a line L maps distinct points to distinct points, i.e., if **A** ≠ **B**, then M(**A**) ≠ M(**B**). Equivalently, it has the property that, for points **A** and **B** in the plane, M(A) = M(B), then **A** = **B**. Thus, reflection is **a one-to-one mapping.**
3. For every point **A'** in the plane, there exists a point **A** such that M(**A**) = **A'**. If the point **A'** is on L, then there exists **A** = **A'** such that M(**A**) = **A'**. Thus, reflection is **an onto mapping.**

Now, consider reflections with respect to the line $y = mx$ and the line $y = mx + b$ for $b \neq 0$.

A. Reflection in the line y = mx

Let L be a line passing through the origin and making an angle θ with the positive x-axis.

You will now find the image of a point P (x, y) when it is reflected about the line L as shown in Figure 6.8.

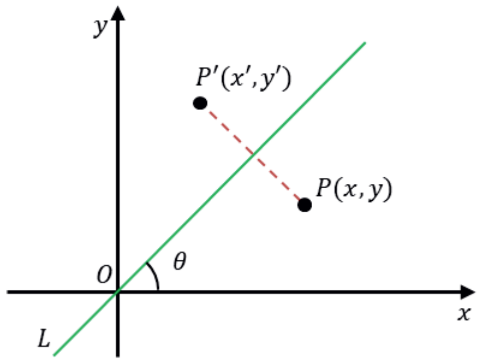


Figure 6.8

You will have the following four special cases based on the value of θ:

> **1. When θ = 0, you will have reflection in the x- axis. Thus, (x, y) is mapped to $(x, -y)$**

Example 1

Find a reflection of a point P (2, 3) over the x-axis.

Solution

As shown in the Figure 6.9, the reflection is P' (2, −3).

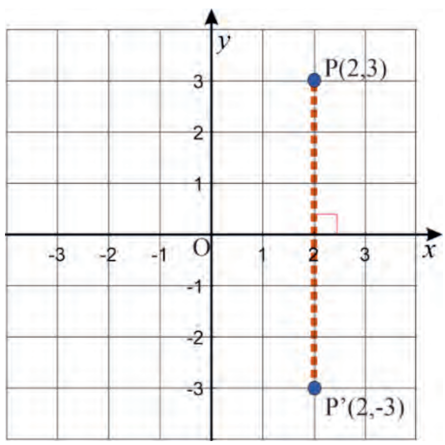


Figure 6.9

2. **When $\theta = \frac{\pi}{2}$, you will have reflection in the y - axis and (x, y) is mapped to $(-x, y)$.**

Example 2

Find a reflection of a point P (2, 3) over the y – axis.

Solution

As shown in the Figure 6.10, the reflection is p'(−2, 3).

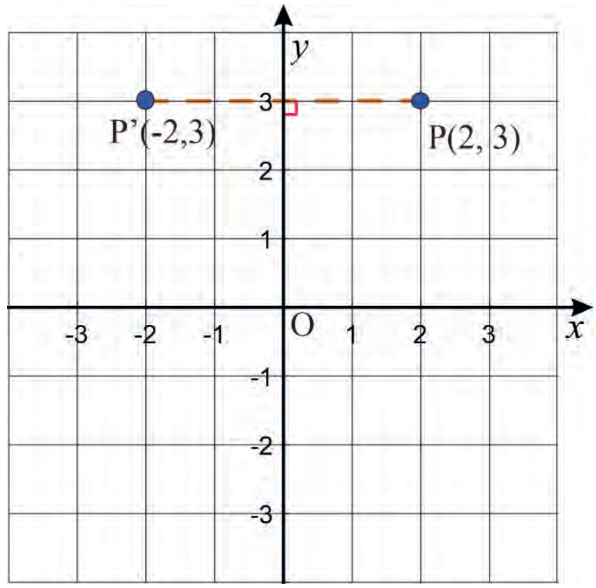


Figure 6.10

3. **When $\theta = \frac{\pi}{4}$, you will have reflection about the line $y = x$ and hence (x, y) is mapped to (y, x).**

Example 3

Find a reflection of a point P (−3, 2) over the line $y = x$.

Solution

As shown in the Figure 6.11, the reflection *is* P' (2, −3).

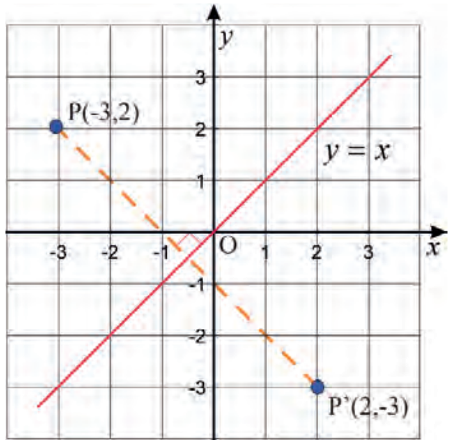


Figure 6.11

4. When θ = $\frac{3\pi}{4}$, you will have reflection about the line $y = -x$ and (x, y) is mapped to $(-y, -x)$.

Example 4

Find a reflection of a point P (2, 3) over the line $y = -x$.

Solution

As shown in the Figure 6.12, the reflection is P'(−3, −2)

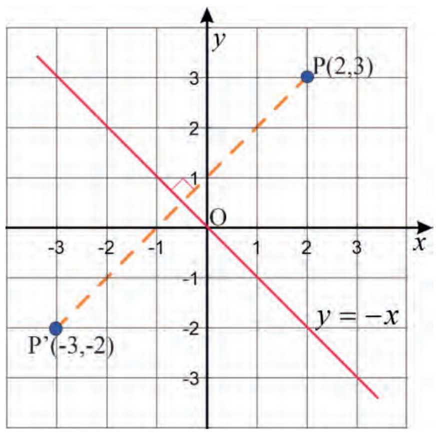


Figure 6.12

In general, reflection of points by the lines $y = 0, x = 0, y = x$ and $y = -x$ using coordinate rules for reflection are:

1. If (x, y) is reflected in the x-axis $(y = 0)$, then its image is the point $(x, -y)$.
2. If (x, y) is reflected in the y-axis $(x = 0)$, then its image is the point $(-x, y)$.
3. If (x, y) is reflected in the line $y = x$, then its image is the point (y, x).
4. If (x, y) is reflected in the line $y = -x$, then its image is the point $(-y, -x)$.

Exercise 6.4

1. Find a reflection of a point P (4, 5) over the following.

   a. x-axis. b. y-axis c. the line $y = x$ d. the line $y = -x$

2. The vertices of △DEF are D (1, 4), E (5, 5), and F (4, 1). Find a reflection Graph △ DEF over the following.

   a. x-axis. b. y-axis c. the line $y = x$ d. the line $y = -x$

Let L be a line passing through the origin and making an angle θ with the positive x-axis.

Then, the slope of L is given by $m = \tan\theta$ and its equation is $y = mx$ as shown in Figure 6.13.

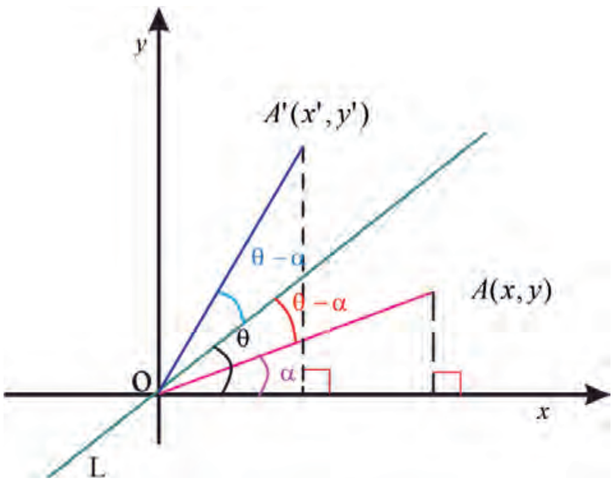


Figure 6.13

You will now find the image of a point A (x, y) when it is reflected about the line L as shown in Figure 6.13.

Suppose α is the angle measured from positive x-axis counter clockwise to the vector $\overrightarrow{OA}$ and A'(x', y') be the image of A (x, y), and let OA = r, then $OA' = r$ (why?).

The coordinates of A are:

$$x = r\cos\alpha \text{ and } y = r\sin\alpha$$

The angle measured from L to vector $\overrightarrow{OA'}$ is $\theta - \alpha$. Thus, the angle measured from positive x-axis to the vector $\overrightarrow{OA'}$ is

$$\theta + (\theta - \alpha) = 2\theta - \alpha$$

Therefore, the coordinates of A' are:

$$x' = r\cos(2\theta - \alpha) \text{ and } y' = r\sin(2\theta - \alpha)$$

Now, use the following trigonometric identities

1. Sine of the sum and the difference

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

2. Cosine of the sum and difference

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Using these trigonometric identities, you obtain:

$x' = r[\cos 2\theta\cos\alpha + \sin 2\theta\sin\alpha] = (r\cos\alpha)\cos 2\theta + (r\sin\alpha)\sin 2\theta.$

$= x\cos 2\theta + y\sin 2\theta$ and

$y' = r[\sin 2\theta\cos\alpha - \cos 2\theta\sin\alpha] = (r\cos\alpha)\sin 2\theta - (r\sin\alpha)\cos 2\theta$,

$= x\sin 2\theta - y\cos 2\theta$

Therefore, the coordinates of P'(x', y') is the image of the point P (x, y) when reflected about the line $y = mx$ is:

$x' = x\cos 2\theta + y\sin 2\theta$

$y' = x\sin 2\theta - y\cos 2\theta$ where θ is the angle of inclination of the line L: $y = mx$.

Example 5

Find the image of the points A (2, 1), B (0, −3) and C (−6, 8) when reflected about the line $y = \frac{1}{\sqrt{3}}x$.

Solution

Since $\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \Rightarrow \theta = \frac{\pi}{6}$

Thus, if A' (x', y') is the image of A (2,1), then

$x' = x\cos 2\theta + y\sin 2\theta = 2\cos 2(\frac{\pi}{6}) + 1\sin 2\left(\frac{\pi}{6}\right) = 2\cos\frac{\pi}{3} + 1\sin\frac{\pi}{3}$

$= 2 \times \frac{1}{2} + 1 \times \frac{\sqrt{3}}{2} = \frac{2+\sqrt{3}}{2}$

$y' = x\sin 2\theta - y\cos 2\theta = 2\sin 2(\frac{\pi}{6}) - 1\cos 2(\frac{\pi}{6}) = 2\sin\frac{\pi}{3} - 1\cos\frac{\pi}{3} = 2 \times \frac{\sqrt{3}}{2} - 1 \times \frac{1}{2}$

$= \frac{2\sqrt{3}-1}{2}$

Therefore, the images of A (2, 1) is A'$\left(\frac{2+\sqrt{3}}{2}, \frac{2\sqrt{3}-1}{2}\right)$

Again, if B' (x', y') is the image of B (0, -3), then

$x' = x\cos 2\theta + y\sin 2\theta = 0 \times \cos\frac{\pi}{3} + (-3)\sin\frac{\pi}{3} = 0 \times \frac{1}{2} - 3 \times \frac{\sqrt{3}}{2} = \frac{-3\sqrt{3}}{2}$

$y' = x\sin 2\theta - y\cos 2\theta = 0 \times \sin\frac{\pi}{3} - (-3)\cos\frac{\pi}{3} = 0 \times \frac{\sqrt{3}}{2} + 3 \times \frac{1}{2} = \frac{3}{2}$

Thus, the image of B (0, –3) is B' $\left(\frac{-3\sqrt{3}}{2}, \frac{3}{2}\right)$.

Similarly, if C' (x', y') is the image of C (−6, 8), then

$x' = x\cos 2\theta + y\sin 2\theta = -6 \times \cos\frac{\pi}{3} + 8\sin\frac{\pi}{3} = -6 \times \frac{1}{2} + 8 \times \frac{\sqrt{3}}{2} = -3 + 4\sqrt{3}$

$y' = x\sin 2\theta - y\cos 2\theta = -6 \times \sin\frac{\pi}{3} - 8\cos\frac{\pi}{3} = -6 \times \frac{\sqrt{3}}{2} - 8 \times \frac{1}{2} = -3\sqrt{3} - 4$

Thus, the image of C (−6, 8) is C' $(-3 + 4\sqrt{3}, -3\sqrt{3} - 4)$.

Exercise 6.5

1. Find the image of the point (0, 1) when reflected about the line $y = \sqrt{3}x$.
2. Find the image of the point (2, 3) when reflected about the line $y = -\sqrt{3}x$.
3. Find the image of the point (3, 0) when reflected about the line $y = 2x$.

B. Reflection in the line *y* = *mx* + *b*

1. Reflection of a point in the line

Let L: y = $mx + b$ be the line of reflection, where $m\in\mathbb{R}\setminus\{0\}$.

Let A (x, y) be a point on the plane, not on L.

Let A'(x', y') be the image of A (x, y) when reflected about the line L.

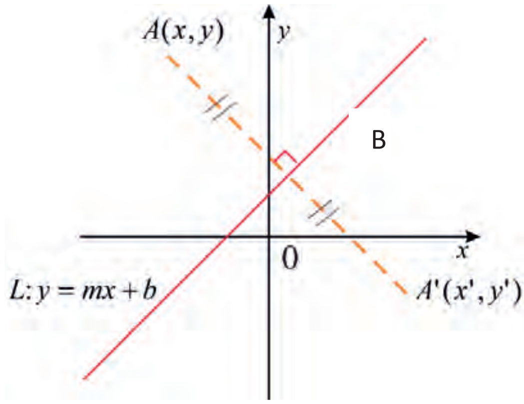


Figure 6.14

Let s be the line passing through the points A (x, y) and A' (x', y'), $\overline{AA'}$ is perpendicular to L, since L is perpendicular bisector of $\overline{AA'}$. Since the slope of L is m, the slope of $\overline{AA'}$ is $-\frac{1}{m}$.

Then, one can determine the equation of the line $\overline{AA'}$. If B is the point of intersection of L and $\overline{AA'}$, taking B as the mid-point of $\overline{AA'}$, you can find the coordinates of A'.

Thus, to find the image of a point A (x, y) when reflected about a line L, you follow the following four steps.

1. Find the slope of the line L, say m.
2. 2. Find the equation of the line $\overline{AA'}$, which passes through the point A (x, y) and has slope $-\frac{1}{m}$.
3. Find the point of intersection B of L and $\overline{AA'}$ which serves as the midpoint of $\overline{AA'}$
4. Using B as the mid-point of $\overline{AA'}$, find the coordinates of A' .

Example 6

Find the image of the point (4, 0) when reflected about the line L: $y = 2x + 1$.

Solution

Let A (4, 0) be a point on the plane, not on L.

Let A'(x', y') be the image of A (4, 0) when reflected about the line L.

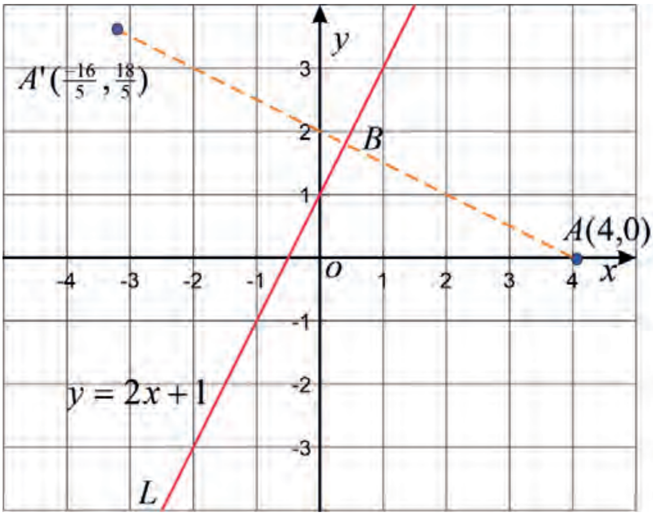


Figure 6.15

Let s be the line passing through the points A $(4, 0)$ and A' (x', y'). $\overline{AA'}$ is perpendicular to L since L is perpendicular bisector of $\overline{AA'}$ and the slope of L is 2. Thus, slope of $\overline{AA'}$ is $-\frac{1}{2}$.

The equation of the line $\overline{AA'}$, which passes through the point A $(4, 0)$ and has slope $-\frac{1}{2}$ is $\frac{y-0}{x-4} = -\frac{1}{2} \Rightarrow y = -\frac{1}{2}x + 2$

Now, let us find the point of intersection B of L and $\overline{AA'}$ which serves as the midpoint of $\overline{AA'}$

Then, B is obtained by solving the system of linear equations $\begin{cases} y = -\frac{1}{2}x + 2 \\ y = 2x + 1 \end{cases}$.

Thus, $(\frac{2}{5}, \frac{9}{5})$ is its solution.

Using B as the midpoint, the image of A (4, 0) can be obtained by using the two points B $(\frac{2}{5}, \frac{9}{5})$ and A (4, 0).

Then, the image of the point A (4, 0) is:

$$\frac{x'+4}{2} = \frac{2}{5} \Rightarrow x' = \frac{-16}{5} \text{ and } \frac{y'+0}{2} = \frac{9}{5} \Rightarrow y' = \frac{18}{5}$$

Thus, the image of A (4, 0) is A'$(\frac{-16}{5}, \frac{18}{5})$.

Example 7

Find the image of (–2, 4) when reflected about the lines

a. $y = -2$ **b.** $x = 1$

Solution

a. Line $y = -2$ is a straight line parallel to x-axis and at a distance of 2 units in negative direction of y-axis. Point $(-2,4)$ is 6 units away from given line $y = -2$. Therefore, distance of image from the line is also 6 units. Thus, distance of image from x-axis = 6 + 2= 8 units in negative direction of y-axis. Thus, the image of the point (–2, 4) when reflected about the line $y = -2$ is (–2, –8).

b. Line $x = 1$ is a straight line parallel to y-axis and at a distance of 1 unit in positive direction of x-axis. Point $(-2,4)$ is 3 unit away from given line $x = 1$. Therefore, distance of image from the line is also 3 units. Thus, distance of image from y-axis = 3 +1= 4 units in positive direction of x-axis. Thus, the image of the point (–2, 4) when reflected about the line $x = 1$ is (4, 4).

Exercise 6.6

1. Find the images of the points (4, 2) when reflected about the line $y = 2x - 1$.
2. Find the images of the points $(-3, -2)$ when reflected about the line $y = -x - 2$.

3. Find the image of (2, 4) when reflected about the lines

   a. $y = 1$ b. $y = -2$

4. If the image of the point (–5, 4) under reflection is (3, 0), find the line of reflection.

5. If P is the reflection of point (4, − 4) in the line $2y = x + 1$, find the coordinates of point P.

1. Reflection of a line in the line

Note

To find the images of a line s when reflected about the line L, take the following way.

1. If a line $\overline{AA'}$ is perpendicular to the axis of reflection L, then $\overline{AA'}$ is its own image.
2. If $\overline{AA'}$ is a line parallel to the line of reflection **L**, to find the image of $\overline{AA'}$ when reflected about L, we follow the following steps.
   a. Choose any point B on $\overline{AA'}$.
   b. Find the image of B, M(B) = B'
   c. Find the equation of $\overline{AA'}$ which is the line passing through B' with slope equal to the slope of L.

Example 8

Find the image of the line s: $-3x + y = 2$ after reflection in the line L: $y = x + 4$.

Solution

Let s: $y = 3x + 2$ and s' be its image after reflecting about the line L: $y = x+4$.

Let A (a, b) be any point on s, say A (0, 2), so that its image $A'(a', b')$ lies on s' as shown in figure 6.16.

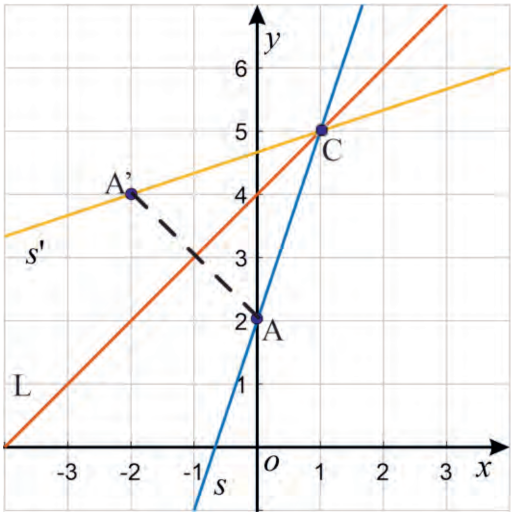


Figure 6.16

By using the following reflecting procedure:

a. Choose any point A on s.

b. Find the image of A, $M(A) = A'$

c. Find the intersection of L and s as C.

d. Find the line passing through both A' and C. This line is the image s'

$M(0, 2) = (a', b') \Rightarrow \frac{b' - 2}{a' - 0} = -1$

$\Rightarrow a' = -b' + 2.$

The midpoint of (0, 2) and (a', b') which is $\left(\frac{a'}{2}, \frac{b' + 2}{2}\right)$ lies on the reflecting axis L.

Then, from

L: $y = x + 4$, it follows

$\Rightarrow \frac{b' + 2}{2} = \frac{a'}{2} + 4.$

$\Rightarrow$ So, $a' = b' - 6.$

Since $a' = -b' + 2$ and $a' = b' - 6$, then

$-b' + 2 = b' - 6$

It implies $b' = 4$ and $a' = -b' + 2 = -4 + 2 = -2.$

Therefore, A' (a', b') = $A'(-2, 4)$.

It follows that $(-2, 4)$ lies on s'.

Now, let us find a point that belongs to both L and s, say C. Then C is obtained by solving the system of equations $\begin{cases} y = x + 4 \\ y = 3x + 2 \end{cases}$.

Thus, (1, 5) is its solution.

Finally, the equation of the image line can be obtained by using the two points of $A'(-2, 4)$ and C (1, 5).

Then, the equation of the imaged line is:

s': $\frac{y-4}{x-(-2)} = \frac{5-4}{1-(-2)}$.

or s': $\frac{y-4}{x+2} = \frac{1}{3}$.

or s': $x - 3y + 14 = 0$.

Thus, the image of line $-3x + y = 2$ after reflecting about the line $y = x + 4$ is s':

$x - 3y + 14 = 0$.

Exercise 6.7

1. Find the image of the line s: $y = 2x + 3$ after it has been reflected about the line L: $y = x - 2$.
2. What's the equation of the image of a line s: $y = 3x + 4$ when it is reflected along the line $y = 2x - 3$?

2. Reflection of a circle in the line

Note

1. If the centre of a circle C is on the line of reflection L, then the image of C is itself.
2. If the centre O of a circle C has image O' when reflected about a line L, then the image circle has centre O' and radius the same as C.

Example 9

Find image of the circle $x^2 + y^2 - 4x - 2y + 4 = 0$ after reflection in the line L: $y = x - 1$.

Solution

By applying completing square method on $x^2 + y^2 - 4x - 2y + 4 = 0$,
you find $(x - 2)^2 + (y - 1)^2 = 1$.
Clearly, the centre of the circle is (2, 1) and it lies on the line of reflection $y = x - 1$
Therefore, the circle is its own image.

Example 10

Given the equation of the circle $x^2 + (y - 3)^2 = 1$, find the equation of the circle after a reflection about the line $y = x$.

Solution

The centre of the circle is (0, 3). The reflection of (0, 3) about the line $y = x$ is (3, 0), which is the centre of the image circle. Therefore, the equation of the image circle is $(x - 3)^2 + y^2 = 1$.

Exercise 6.8

1. Find the image of the circle $x^2 + y^2 = 4$ after reflection in the line $L: y = x$.
2. Find the image of the circle $(x - 4)^2 + (y - 1)^2 = 9$ after reflection in the line $L: y = x - 1$.
3. Given an equation of a circle $(x - 1)^2 + (y - 2)^2 = 36$, find the equation of the image circle after a reflection about the line $y = x + 2$.

4. The image of the circle $x^2 + y^2 - x + 2y = 0$ when it is reflected about the line L is $x^2 + y^2 - 2x + y = 0$. Find the equation of L.

6.4 Rotation

Activity 6.4

1. Discuss rotation of points through 90 ° and 180° about the origin.
2. Consider a triangle ABC, discuss what happened the shape, size and orientation of a triangle when you turn counter clockwise it around a fixed point as shown in Figure 6.17 and try to define what does rotation of figures mean.

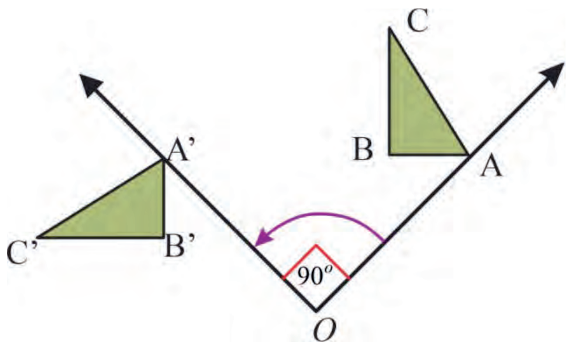


Figure 6.17

As you have observed form Figure 6.17, when Δ ABC is rotated about a point O or the centre of rotation to be the origin, Δ ABC and Δ $A'B'C'$ have the same shape and size.

Definition 6.4

A rotation R about a point O through an angle θ is a transformation of the plane onto itself which carries every point A of the plane into the point A' of the plane such that $OA = OA'$ and m ($\angle AOA'$) = θ. O is called the centre of rotation and θ is called the angle of rotation.

A. Rotation when the center of rotation is about the origin

Note

1. The rotation is in the counter clockwise direction, if $\theta > 0$ and in the clockwise direction if $\theta < 0$.
2. Rotation is a rigid motion but the figures may be turned in different directions.

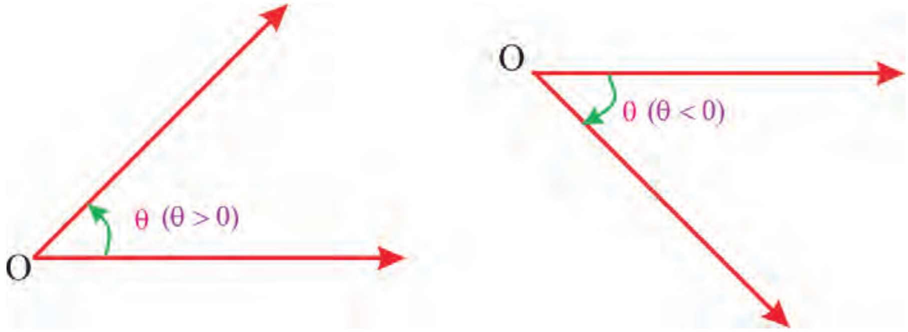


Figure 6.18

Example 1

Find the image of point A (2, 0) when it is rotated through 30° about the origin.

Solution

Let the image of the point A (2, 0) be A'(a, b) as shown in the Figure 6.19.

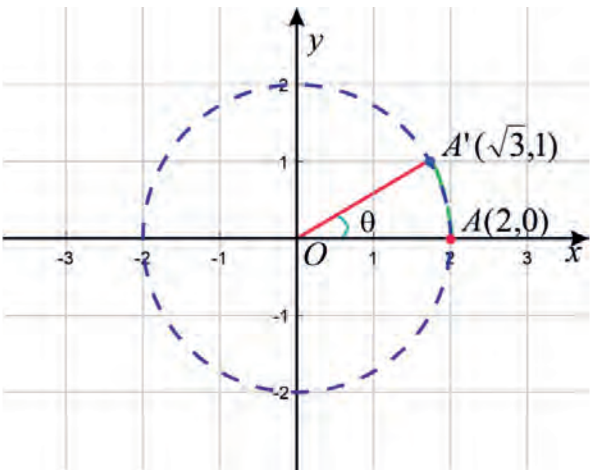


Figure 6.19

From trigonometry, you have $A'(a, b) = (r\cos\theta, r\sin\theta)$ where $r = 2$ and $\theta = 30°$

Therefore, A'(a, b) = (2cos30°, 2sin30°).

Hence, A' $(a, b) = (2\,(\frac{\sqrt{3}}{2}), 2\,(\frac{1}{2})) = (\sqrt{3}, 1)$.

Thus, the image of A (2, 0) is A'$(\sqrt{3},\ 1)$.

Notation

If R is rotation through an angle θ, then the image of A (x, y) is denoted by $R_\theta\,(x, y)$. For example, in the above example, $R_{30°}\,(2, 0) = (\sqrt{3}, 1)$

Exercise 6.9

Find the image of point P (3, 0) when it is rotated through the following angle about the origin:

a. $\theta = 45°$ b. $\theta = 60°$ c. $\theta = 120°$ d. $\theta = 135°$

e. $\theta = 150°$ f. $\theta = -45°$ g. $\theta = -60°$ h. $\theta = -120°$

i. $\theta = -135°$ j. $\theta = -150°$

Rotation when the center of rotation is about the origin (1)

Theorem 6.1

Let R be a rotation through angle θ about the origin. If $R_\theta\,(x, y) = (x', y')$, then

$x' = x\cos\theta - y\sin\theta$ $y' = x\sin\theta + y\cos\theta$

Proof

From trigonometry, you have,

$(x, y) = (r\cos\alpha, r\sin\alpha)$ and $(x', y') = (r\cos(\alpha+\theta), r\sin(\alpha+\theta))$,

where $r\cos(\alpha+\theta) = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$… (cosine of the sum formula)

$= x\cos\theta - y\sin\theta$…………because $x = r\cos\alpha$ and $y = r\sin\alpha$

and $r\sin(\alpha+\theta) = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta$……… (sine of the sum formula)

$= y\cos\theta + x\sin\theta$…………because $x = r\cos\alpha$ and $y = r\sin\alpha$

$\therefore R_\theta(x, y) = (x \cos\theta - y \sin\theta, y \cos\theta + x \sin\theta)$

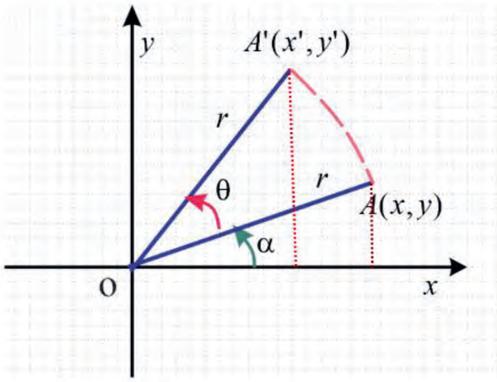


Figure 6.20

Example 2

Find the images of the following points under a rotation by the indicated angle about the origin.

a. (1, 3), 30° b. (3, − 4), 810°

Solution

From theorem 6.1, you have

a. $x' = x\cos\theta - y\sin\theta$; $x = 1, y = 3$; $\theta = 30°$

$= 1\cos 30° - 3 \times \sin 30°$

$= 1 \times \frac{\sqrt{3}}{2} - 3 \times \frac{1}{2} = \frac{\sqrt{3}-3}{2}$

$y' = x\sin\theta + y\cos\theta$

$= 1\sin 30° + 3\times\cos 30°$

$= \frac{1}{2} + 3\times\frac{\sqrt{3}}{2} = \frac{1+3\sqrt{3}}{2}$

Thus, $R_{30°}(1, 3) = \left(\frac{\sqrt{3}-3}{2}, \frac{1+3\sqrt{3}}{2}\right)$.

b. A rotation of 810 ° is the same as two consecutive rotations by 360 ° followed by a rotation by 90° (because 810 ° = 2×360 ° + 90°).

Hence, a rotation by 810° is the same as a rotation by 90°. Thus, we can simply use theorem 6.1 for $\theta = 90°$

$x' = x\cos\theta - y\sin\theta;\ x = 3, y = -4;\ \theta = 90°$

$= 3 \times \cos(90°) - (-4) \times \sin(90°)$

$= 3 \times 0 + 4 \times 1 = 4$

$y' = x\sin\theta + y\cos\theta$

$= 3 \times \sin(90°) + (-4) \times \cos(90°)$

$= 3 \times 1 - 4 \times 0 = 3$

Thus, $R_{810°}(3, -4) = R_{90°}(3, 4) = (4, 3)$.

Exercise 6.10

1. Find the images of the following points under a rotation by the indicated angle about the origin:

   a. $(2, 4), 30°$ b. $(4, -5), \frac{\pi}{2}$ c. $(5, -6), \frac{\pi}{2}$ d. $(2, 1), -45°$

2. Determine the images of the following points after a rotation of the indicated angle about the origin:

   a. $(-2, 4), -60°$ b. $(5, -6), -180°$ c. $(2, -2), -\frac{3\pi}{2}$ d. $(3, 4), -540°$

Rotation when the center of rotation is about the origin (2)

Note

1. $\theta = \frac{\pi}{2}$

Let R be a counter-clockwise rotation through an angle θ about the origin. Then $\theta = \frac{\pi}{2} \Rightarrow (x', y') = (-y, x)$. This means when $\theta = \frac{\pi}{2}$,

$$x' = x\cos\theta - y\sin\theta = -y$$

$$y' = x\sin\theta + y\cos\theta = x$$

Example 3

The image of a triangle ABC when it is rotated through $\theta = \frac{\pi}{2}$ or $90°$ about the origin is shown in Figure 6.21.

To see that this is a rotation of $\theta = \frac{\pi}{2}$, imagine point B attached to the red arrow. The red arrow is then moved $\theta = \frac{\pi}{2}$ (Notice that the $\frac{\pi}{2}$ or $90°$ is formed by the two red arrows).

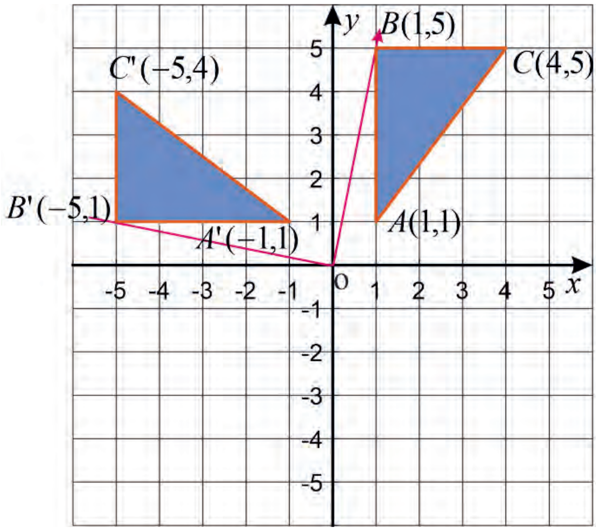


Figure 6.21

Look at the new position of B, labelled $B' = (-5,1)$. This same approach can be used for all three vertices.

2. $\theta = \pi \Rightarrow R(x,y) = (-x,-y)$.

Example 4

The image of a triangle ABC when it is rotated through $\theta = \pi$ or $180°$ about the origin is shown in Figure 6.22.

Starting with ΔABC, draw the rotation of 180º centred at the origin. As you did in the previous example, imagine point B attached to the red arrow from the center (0, 0).

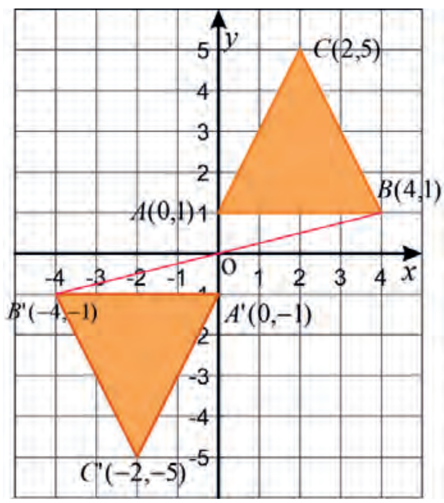


Figure 6.22

The arrow is then moved θ = π or 180º (which forms a straight line). Notice the new position of B, labeled B' = (−4, −1).

3. $\theta = \frac{3\pi}{2} \Rightarrow R(x, y) = (y, -x)$.

Example 5

The image of a quadrilateral ABCD when it is rotated through $\theta = \frac{3\pi}{2}$ or 270 ° about the origin is shown in Figure 6.23.

Starting with quadrilateral *ABCD*, draw the rotation of 270º centred at the origin. As you did in the previous examples, imagine point *A* attached to the red arrow from the centre (0, 0). The arrow is then moved 270º. Notice the new position of *A*, labelled $A' = (1, 0)$.

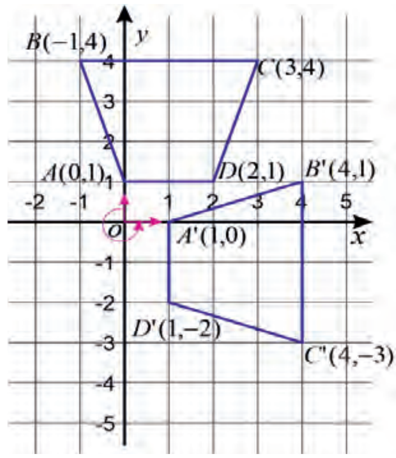


Figure 6.23

4. $\theta = 2n\pi$ for $n \in Z \Rightarrow$ R is the identity transformation.

5. Every circle with centre at the centre of rotation is fixed.

In general, you can summarize rotation of point centred at the origin as follows

Rotation of 90°	(x, y) becomes $(-y, x)$
Rotation of 180°	(x, y) becomes $(-x, -y)$
Rotation of 270°	(x, y) becomes $(y, -x)$

Exercise 6.11

1. A triangle CDE has vertices C = (2, 1), D= (5, 3) and E = (3, 4). Find the image of the triangle when rotated about the origin through an angle $\theta = \frac{\pi}{2}$.

2. Determine the image of the straight-line AB under an anticlockwise rotation of 90° about O in the Figure 6.24.
3. If A (2, −2), B (3, −1) and C (−6, 1) are the vertices of a triangle ABC as shown in Figure 6.25, determine the image of the triangle ABC under a clockwise rotation of 180° about the origin.

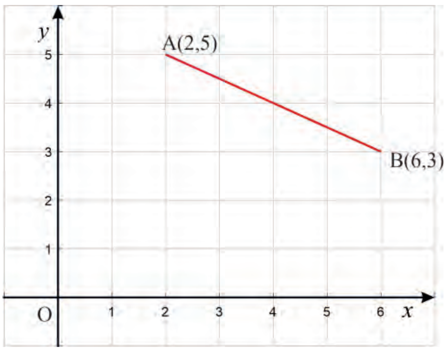


Figure 6.24

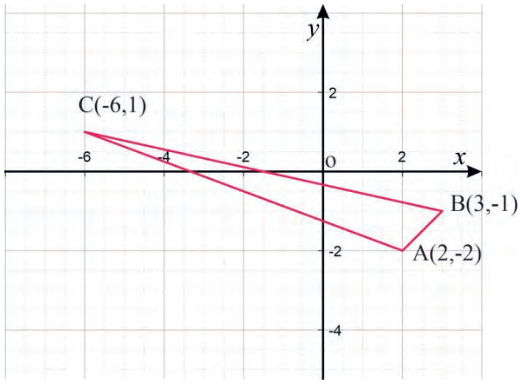


Figure 6.25

4. Given an equation of a circle $(x + 1)^2 + (y + 3)^2 = 4$, find the image circle under an anticlockwise rotation of 180° about the origin.

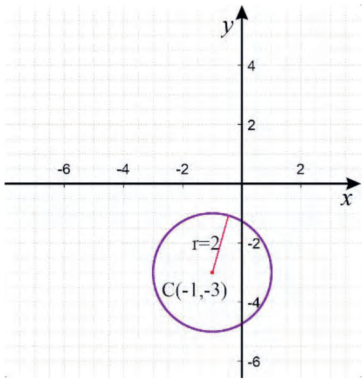


Figure 6.26

B. Rotation when the center of rotation is the point (a, b)

In previous section, you saw rotation about the origin. In this section you will discuss rotation about an arbitrary point (a, b).

Consider the Figure 6.27 as shown below.

It indicates how the original triangle ABC that has vertices A (4,7), B (2,7) and C (4, 2) is rotated through 180° about the point (4, 1) to give triangle $A'B'C'$.

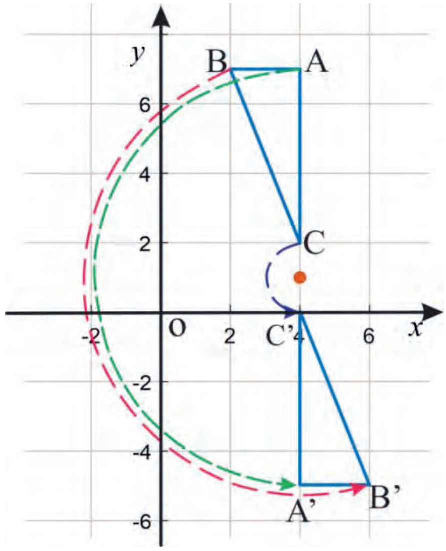


Figure 6.27

As you can observed from Figure 6.27, you can generalize the following formula:

Corollary 6.1

Rotation about the point (a, b) can be divided the following three transformation.

1. Translation by $T = (-a, \ -b)$

$$(x_1', \ y_1') = (x - a, \quad y - b)$$

→ The center of rotation is moved to the origin

2. Rotation through an angle θ about the origin

$$x_2' = x_1' \cos \theta - y_1' \sin \theta$$

$$y_2' = x_1' \sin \theta + y_1' \cos \theta$$

3. Translation by $T = (a, \ b)$

$$x_3' = a + x_2' = a + x_1' \cos \theta - y_1' \sin \theta$$

$$= a + (x - a) \cos \theta - (y - b) \sin \theta$$

$$y_3' = b + y_2' = b + x_1' \sin \theta + y_1' \cos \theta$$

$$= b + (x - a) \sin \theta + (y - b) \cos \theta$$

Example 6

Find the image of the point A (3, 4) when it is rotated through $\theta = \pi$ about (6, 5).

Solution

From the corollary 6.1, you have,

$$x' = a + (x - a)\cos\theta - (y - b)\sin\theta$$

$$y' = b + (x - a)\sin\theta + (y - b)\cos\theta$$

where $(x, y) = (3, 4)$, $(a, b) = (6, 5)$, $\theta = \pi$

$$x' = 6 + (3 - 6)\cos\pi - (4 - 5)\sin\pi$$

$$= 6 + (-3)(-1) - (-1)(0)$$

$$= 9$$

$$y' = 5 + (3 - 6)\sin\pi + (4 - 5)\cos\pi$$

$$= 5 + (-3)(0) + (-1)(-1)$$

$$= 6$$

Thus, the image of the point A (3, 4) after rotating 180° about (6, 5) is (9, 6).

Note

One can also obtain the image of a line under a given rotation as follows:

1. Choose two points on the line.
2. Find the images of the two points under the given rotation.

Thus, the image line will be the line passing through the two image points.

Example 7

Determine the image of the line $L: y = 3x + 1$ after a rotation of $\theta = \frac{\pi}{2}$ about the point (2, 2).

Solution

From the above note, you choose any two arbitrary points, say $(x_1, y_1) = (1, 4)$ and $(x_2, y_2) = (-2, -5)$ as shown in figure 6.28

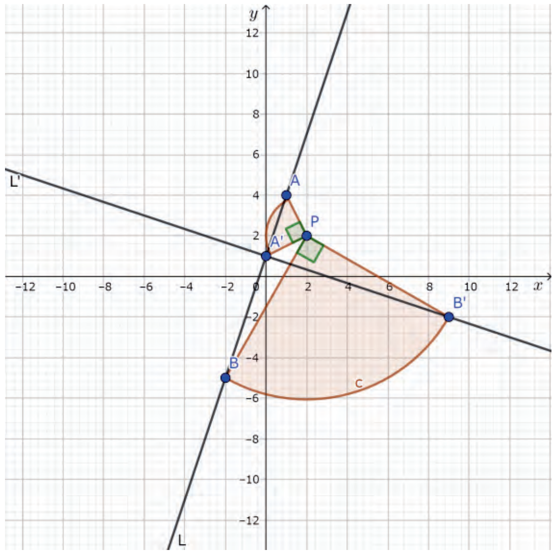


Figure 6.28

But from corollary 6.1, you have

$$x' = a + (x - a)\cos\theta - (y - b)\sin\theta$$

$$y' = b + (x - a)\sin\theta + (y - b)\cos\theta$$

where $(x, y) = (x_1, y_1) = (1, 4)$ or $(x_2, y_2) = (-2, -5)$, $(a, b) = (2, 2)$ and $\theta = \frac{\pi}{2}$

Thus, $P(x_1, y_1) = (a + (x_1 - a)\cos\theta - (y_1 - b)\sin\theta,\ b + (x_1 - a)\sin\theta + (y_1 - b)\cos\theta)$

$$P(1, 4) = (2 + (1 - 2)\cos\frac{\pi}{2} - (4 - 2)\sin\frac{\pi}{2},\ 2 + (1 - 2)\sin\frac{\pi}{2} + (4 - 2)\cos\frac{\pi}{2})$$

$$= (2 - 0 - 2(1),\ 2 - (1) + 2(0))$$

$$= (0, 1)$$

$P(x_2, y_2) = (a + (x_2 - a)\cos\theta - (y_2 - b)\sin\theta,\ b + (x_2 - a)\sin\theta + (y_2 - b)\cos\theta)$

$$P(-2, -5) = (2 + (-2 - 2)\cos\frac{\pi}{2} - (-5 - 2)\sin\frac{\pi}{2},\ 2 + (-2 - 2)\sin\frac{\pi}{2} + (-5 - 2)\cos\frac{\pi}{2})$$

$$= (2 + ((-2 - 2)0) - ((-5 - 2)\,1),\ 2 + ((-2 - 2)1) + ((-5 - 2)\,0))$$

$$= (9, -2)$$

Now, let L' be the imaged line containing the points (0,1) and (9, −2).

Thus, slope of L' $= \frac{-2-1}{9-0} = -\frac{3}{9} = -\frac{1}{3}$

$$L': \frac{y-1}{x-0} = \frac{-1}{3}$$

$$\Rightarrow L': y = -\frac{1}{3}x + 1.$$

Note

As in the case of translation and reflection, to find the image of a circle under a given rotation we follow the following steps:

1. Find the centre and radius of the given circle.
2. Find the image of the centre of the circle under the given rotation.
3. Equation of the image circle will be an equation of the circle centred at the image of the centre of the given circle with radius the same as the radius of the given circle.

Example 8

Find the image of the circle $(x+4)^2 + (y-6)^2 = 9$ when it is rotated through $\theta = \pi$ about (−7, −3).

Solution

From the above note, you compute only the image of the centre of the circle.

The centre is (−4, 6) and its radius is 3

But from corollary 6.1, you have

$$x' = a + (x-a)\cos\theta - (y-b)\sin\theta$$

$$y' = b + (x-a)\sin\theta + (y-b)\cos\theta$$

where $(x, y) = (-4, 6)$, $(a, b) = (-7, -3)$, $\theta = \pi$

$$x' = -7 + (-4-(-7))\cos\pi - (6-(-3))\sin\pi$$

$$= -7 + 3(-1) - 9(0)$$

$$= -10$$

$y' = -3 + (-4 - (-7)) \sin\pi + (6 - (-3)) \cos\pi$

$= -3 + 3(0) + 9(-1)$

$= -12$

Thus, the equation of the image of the circle is $(x + 10)^2 + (y + 12)^2 = 3^2$.

Exercise 6.12

1. Find the image of (0, 1) after it has been rotated 45° about (5, 3).
2. Find an equation of the line into which the line with the given equation is transformed under a rotation through the indicated angle about (0, 1):

   a. $2x - 3y = 5; \theta = \frac{8\pi}{3}$ b. $4x + 5y = 4; \theta = -\frac{\pi}{4}$

3. Find an equation of the circle into which the circle with the given equation is transformed under a rotation through the indicated angle, about the (1, 0):

   a. $x^2 + y^2 = 3, \theta = \frac{3\pi}{2}$ b. $(x + 1)^2 + (y - 2)^2 = 6^2, \theta = \frac{\pi}{3}$

6.5 Applications

In this unit, you discussed definition and properties of transformation (translation, reflection and rotation) with their application in determining the images of a point, a line, a circle and plane figures. Now, you will consider further applications involving transformations.

Example 1

Figure 6.29 shows a reflection in line L.

a. Which two points are invariant?

b. If the length of FG is 14cm, what is the length of GH?

c. If the length of FH is 16 cm, what is the length of EH?

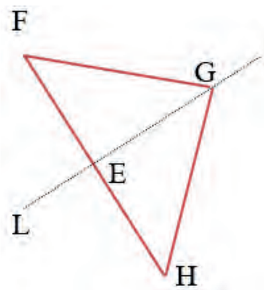


Figure 6.29

Solution

a. The invariant points are E and G.

b. Let $\overrightarrow{FG}$ = 14cm then $\overrightarrow{GH}$ = 14cm. Since, $\overrightarrow{GH} = \overrightarrow{FG}$ = 14cm.

c. If the length of FH = 16cm then the length of

EH = $\frac{1}{2}$(length of FH) = $\frac{1}{2} \times$ 16cm = 8cm.

Example 2

Given the translation $(x + 6, y - 2)$, find the image of the point (2, 6).

Solution

The given translation is, $(x, y) \rightarrow (x + 6, y - 2)$.

It is given that $(x, y) = (2, 6)$. Hence, the translated point is,

$(x + 6, y - 2) = (2 + 6, 6 - 2) = (8, 4)$.

Exercise 6.13

1. Figure 6.30 below shows a reflection in a horizontal axis. Some length in this Figure are AB = 4cm, HI = 3cm and FH = 3cm.

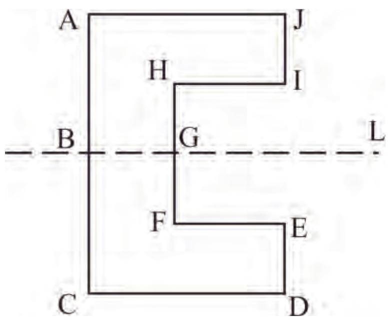


Figure 6.30

a. What point is the image of I?

b. What line segment is the image of CD?

c. Write down any invariant points.

d. What is the length of EF?

e. What is the length of GH?

f. What is the length of AC?

2. A shape is formed with vertices (1, 6), (−3, −3), (−4, 5), and (−6, −4). Plot the image of this shape with respect to the translation $(x, y) \rightarrow (x + 6, y + 1)$.

Problem Solving

1. Let D (−5, 3), E (−3, 0) and F(−5, −2) be the three vertices of a triangle. If this triangle is reflected about the line $x = -2$, what will be the new vertices D', E' and F'?
2. ΔABC with A (1,1), B (7,−2) and C (1,−2) is rotated 90º about point R (−2,−4). What are the coordinates of the image of point B?

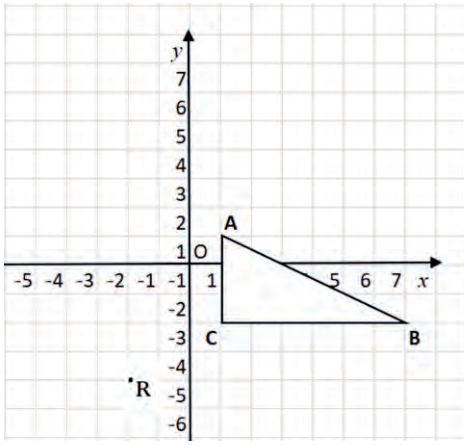


Figure 6.31

3. Triangle ABC has vertices at A (2,3), B (5,7), and C (5,3) in the coordinate plane. The triangle will be reflected over the x-axis and then rotated 180° about the origin to form $\Delta A'B'C'$. What are the vertices of $\Delta A'B'C'$?

Summary

1. Transformation of the plane

a. Transformation can be classified as rigid motion and non-rigid motion.

b. Rigid motion is a motion that preserves distance. Otherwise, it is non-rigid.

c. Identity transformation is a transformation in which image of every point is itself.

2. Translation

A translation is a transformation that occurs when every point of a figure is moved from one location to another location along the same direction through the same distance:

1. If point A is translated to point A', then the vector $\overrightarrow{AA'}$ is called the translation vector.

2. If T = (a, b) is a translation vector, then

   i. The origin is translated to (a, b). That means $(0, 0) \longrightarrow (a, b)$.

   ii. The image of the point A (x, y) under the translation T = $A'(x + a, y + b)$.

3. Reflection

Let L be a fixed line in the plane. A reflection M about a line L is a transformation of the plane onto itself which carries each point A of the plane into the point A' of the plane such that L is the perpendicular bisector of AA'.

a. Reflection in the *x-axis*, M $(x, y) = (x, -y)$.

b. Reflection in the *y-axis*, M $(x, y) = (-x, y)$.

c. Reflection in the line $y = x$, M $(x, y) = (y, x)$.

d. Reflection in the line $y = -x$, M $(x, y) = (-y, -x)$.

e. Reflection in the line $y = mx$, M $(x, y) = (x', y')$ where

   $x' = x \cos 2\theta + y \sin 2\theta$; $y' = x \sin 2\theta - y \cos 2\theta$ and $m = \tan \theta$

4. Rotation

A rotation R about a point O through an angle θ is a transformation of the plane onto itself which carries every point A of the plane into the point A' of the plane such that OA = OA' and m ($<AOA'$) = θ. O is called the centre of rotation and θ is called the angle of rotation.

1. If R is a rotation through angle θ about the origin), then

   $R_\theta\ (x, y) = (x', y') = (x' = x \cos\theta - y \sin\theta,\ y' = x \sin\theta + y \cos\theta)$

2. Let R be a counter-clockwise rotation through an angle θ about the origin. Then

   a. If $\theta = \frac{\pi}{2}$ then $R_\theta(x, y) = (-y, x)$.

   b. If $\theta = \pi$ then $R_\theta\ (x, y) = (-x, y)$.

   c. If $\theta = \frac{3\pi}{2}$ then $R_\theta\ (x, y) = (y, x)$.

   d. If $\theta = 2n\,\pi$ then $R_\theta\ (x, y)$ is the identity transformation.

   e. Every circle with centre at the centre of rotation is fixed.

Review Exercise

1. If a translation T takes the point (3, −5) to the point (1, 4), then find the images of the following line and circle.
   a. $5x - 3y = 5$ b. $y + 3x = 6$
   c. $x + 2y = 7$ d. $x^2 + y^2 - 4x + 6y = 8$
   e. $x^2 + y^2 = 4$ f. $x^2 + y^2 - 3x + 5y = 0$
2. The image of the point A (4, 9) in a reflection is A' (5, 1). Find the line of reflection's equation.
3. Find the image of the circle $x^2 + y^2 - 6y = 3$ when it is rotated through $\frac{\pi}{3}$ about (−1,5).
4. Rotate triangle ABC if A (5, 1), B (3, 2), C (3, −2) 90° clockwise about the origin, then reflect the image over the line $x = 4$.
5. If the image of the point (–3, 1) under reflection is (0, 2), then find the line of reflection.
6. If the image of the line $3x - 4y = 11$ under a translation is $3x - 4y = 0$. What is the translation vector of the translation line?
7. Find the equation of the image of the circle $x^2 + y^2 + 16x - 24y + 183 = 0$ when it is reflected about the line $4x + 7y + 13 = 0$?
8. If the line with equation $3x + 2y = 1$ is rotated clockwise about the point (−1, 2) through $\theta = 45^\circ$, then find the equation of this new line?
9. Find the image of the circle $(x + 10)^2 + (y + 12)^2 = 9$ when it is rotated through $\theta = 30^\circ$ about (7, 3).
10. Find the image of the line s: $y = x + 5$ after it has been reflected about the line
    a. $L: y = \frac{1}{2}x + 2.$ b. $L: y = x - 1.$
    c. $L: y = \frac{1}{3}x + 1.$ d. $L: y = 2x + 1.$

UNIT 7

STATISTICS

Unit Outcomes

By the end of this unit, you will be able to:

- Know specific facts about types of data.
- Understand basic concepts about grouped data.
- Use statistical method to solve real life problems.
- Analyze statistical data.
- Compare statistical data.
- Appreciate the value of statistics in real life.

Unit Contents

KEY Words

- class interval
- class limit
- class midpoint
- continuous data
- deciles
- discrete data
- frequency
- mean
- measures of location
- mode
- percentiles
- qualitative data
- quantitative data
- quartiles

Introduction

In grade 9, you have seen the meaning, importance and purpose of statistics. Moreover, you have seen presentation of data using different forms like a histogram, measures of central tendency, and measures of dispersion of ungrouped data.

At this grade, you will start types of data but initially through discussion.

Florence Nightingale (1820 – 1910), was a famous nurse during the Crimean War, which took place between 1853 and 1856. She is generally considered to have founded the modern nursing profession, by establishing her own nursing school at St Thomas' Hospital in London in 1860. She was also a gifted mathematician, and used statistical diagrams to illustrate the conditions that existed in the hospitals where she worked. Although she did not invent pie chart, she popularized its use, along with other diagrams such as the rose diagram, which is like a circular histogram. In 1859, she was elected the first female member of the Royal Statistical Society.

Source: Cambridge IGCSE Mathematics Fifth Edition.

7.1 Types of Data

1. Qualitative and Quantitative Data

Activity 7.1

Consider the following data: gender, height, weight, beauty, time, colour, price and temperature. Discuss which of these data can be or cannot be measured/expressed in the form of numbers.

From Activity 7.1 you have observed that there are data that can be measured in a form of number values while there are also data that cannot be expressed in the form of number values. These data can be classified as quantitative or qualitative data.

Definition 7.1

Quantitative data is the type of data whose value is measured in the form of numbers or counts, with a unique numerical value associated with each data set.

Qualitative data is non-numerical data describing the attributes or properties that an object possesses.

Example 1

Classify the following data as qualitative or quantitative: height, weight, the street a person lives on, the car a person drives, income, length, time, distance, religion, social status, number of mathematics teachers in your school and price.

Solution

The street a person lives on, the car a person drives, religion and social status are qualitative, while height, weight, income, length, time, number of mathematics teachers in your school, price and distance are quantitative.

2. Discrete Data and Continuous Data

Activity 7.2

Consider the following quantitative data: height, weight of the students, number of teachers in your school, length of a road, number of chairs in a room, temperature of a body of students and number of houses along a street. Discuss which of these data can be quantified in terms of whole numbers values. Which can be quantified in terms of fractional values?

There are quantitative data that can be quantified in terms of whole number values only or fractional values and varying data values that are measured over a specific time interval. Depending on this criterion, therefore, these data can be classified as discrete or continuous data.

Definition 7.2

A **Discrete data** is a numerical type of data that includes whole, concrete numbers with specific and fixed data values determined by counting. There is a gap between values.

A **Continuous data** is one which takes any fractional point along a specified interval of values.

Example 2

Which of the following are discrete data? Which are continuous?

The number of persons per household, the weight of each shipment of exported coffee, the units of an item in inventory, the length of time between successive landings of airplane at Bole Airport, the number of assembled components which are found to be defective.

Solution

Table 7.1

Discrete data	Continuous data
The number of persons per house hold	The weight of each shipment of exported coffee
The units of an item in inventory	The length of time between successive landings of airplane at Bole Airport
The number of assembled components which are found to be defective	

Definition 7.3

Variables are the characteristics of a unit being observed that may assume more than one of a set of values to which a numerical measure can be assigned and denoted by letters as x, y, z, ...

Example 3

Suppose in your class the height, weight, age of students, eye colour of students, dog breed, climate, electrical conductivity, customer service satisfaction and class attendance vary. Thus, these quantities are some examples of variables.

Exercise 7.1

1. Classify the following data as quantitative and qualitative:

   a. Gender
   b. Religion
   c. Amount of money you have
   d. State of birth
   e. Number of students
   f. Educational qualification
   g. Shoe size
   h. Hair color
   i. Volume of water in a barrel
   j. score of a team in a soccer match

2. Identify whether each of the following is discrete or continuous data:

   a. weight of a package ready to be shipped
   b. rank of students by examination results

c. volume of water in a glass
d. yield of wheat in quintals
e. number of red marbles in a jar
f. time it takes to get to school
g. distance travelled between classes
h. students' grade level
i. year of birth
j. winning time in a race

7.2 Introduction to Grouped Data

Having a collected and edited data, the next important step is to organize it. That is to present it in a readily comprehensible condensed form in order to draw conclusion from it. Hence, before discussing about grouped data let us define a grouped data and frequency distribution of a data as follows.

Definition 7.4

1. **Grouped data are** data formed by aggregating individual observations of a variable into groups.
2. **Frequency distribution** is a list, table or graph that displays the frequency of various outcomes in a sample.

7.2.1 Grouped Discrete Data

Activity 7.3

Consider the following data.

4	5	2	8	8	7	5	5	6	7
6	6	8	8	7	5	7	7	4	4
4	4	1	0	0	6	1	9		

It represents students' results of mathematics test out of ten. Discuss how this data can be organized into three groups of classes.

From Activity 7.3, you can make a frequency distribution table with a group of three classes that help you to define the grouped frequency distribution, grouped discrete data and other related definitions.

Definition 7.5

1. **Grouped discrete data** is a discrete data that has been grouped into categories.
2. **Grouped frequency distribution** is the organization of raw data in table form using classes and frequencies for the purpose of summarizing a large sample of data. When the range of the data is large, the data can be divided into classes with more than one unit in width.
3. **Class interval** is the numerical width of any class in a particular distribution.
4. **Class limit** is the minimum value and the maximum value the class interval may contain.
5. **Class width** is the difference between the upper- and lower-class limit of any class (category).

Note

Steps for constructing grouped frequency distribution

1. Determine the number of classes required (usually between 5 and 20).
2. Approximate the interval of each class or class width using the following formula

$$\text{Class width} = \frac{\text{Largest Value in ungrouped data} - \text{Smallest Value in ungrouped data}}{\text{number of classes required}}$$

Example 1

The following raw data represents the number of people who visited Unity Park per day for 20 days

11	29	6	33	14	31	22	27	19	20
18	17	22	38	23	21	26	34	39	27

a. Construct a grouped frequency distribution with 6 classes.

b. What is the frequency of the 3rd class?

c. What is the frequency of the 6th class?

Solution

a. The data given is raw data, or ungrouped data. Follow these steps to summarize the raw data into a grouped frequency distribution:

Step 1: Determine the number of classes.

To prepare the grouped frequency distribution, you decide the number of classes. For this case, let the number of classes be 6.

Step 2: Find class Width.

From the formula for class width, you have:

$$\text{Class width } (w) = \frac{\text{Largest Value in ungrouped data} - \text{Smallest Value in ungrouped data}}{\text{number of classes required}}$$

$$= \frac{39 - 6}{6} = 5.5$$

From the formula, the class width (w) is calculated as 5.5. For practical purposes, it will be convenient to choose the class width to be a whole number. For this case, you can take class width as 6. (This is obtained by rounding 5.5 to the nearest whole number). Therefore, $w = 6$.

The complete grouped frequency distribution is as follows:

Table 7.2

Number of people (Class interval)	Tally	Number of visiting days (f)
6 – 11	//	2
12 – 17	//	2
18 – 23	~~////~~ //	7
24 – 29	////	4
30 – 35	///	3
36 – 41	//	2
Total		20

b. The frequency of 3^{rd} class is 7.

c. The frequency of 6^{th} class is 2.

Example 2

For the above table, give the lower- and upper-class limits for the 3rd and the 6th classes.

Solution

For the 3rd class 18 – 23, 18 is called the lower-class limit and 23 is called the upper-class limit, while the lower limit and the upper limit of the 6th class are 36 and 41, respectively.

Exercise 7.2

The following data shows the test results of 20 students in mathematics:

1	2	3	3	4	5	5	5	6	6
6	6	6	7	8	8	9	10	10	10

a. Construct a grouped frequency distribution with 5 classes.

b. What is the frequency of the 4th class?

c. What is the frequency of the 2nd class?

d. What is the upper-class limit and lower-class limit of the 3rd class?

Definition 7.6

1. **Cumulative frequency** is the total of a frequency and all frequencies in a frequency distribution until a certain defined class interval.
2. **Cumulative frequency distribution** is a form of frequency distribution that represents frequency distribution.

Example 3

Construct a cumulative frequency distribution for example 1 above.

Solution

In the above grouped frequency distribution, you are considering frequencies of each class. But in reality, you may be interested to know about other issues such as how many days fewer than 24 people visited the Unity Park. To answer such a question, the frequency distribution given above may not always be suitable. For such a purpose, you need to construct what is called a cumulative frequency distribution.

A cumulative frequency distribution is constructed by successively adding the frequencies of each class.

The cumulative frequency distribution of the above data of people that visit Unity Park per day is as follows:

Table 7.3

Number of people (Class interval)	Tally	Number of visiting days(f)	Cumulative frequency (Cf)
6 – 11	//	2	2
12 – 17	//	2	4
18 – 23	~~////~~ //	7	11
24 – 29	////	4	15
30 – 35	///	3	18
36 – 41	//	2	20

Exercise 7.3

1. The following data represents the number of people who paid TAX per day for 30 working days (hint: use excel software to present the ascending order of the data)

1	1	4	4	3	6	5	3	6	5
3	7	7	6	7	8	7	8	8	8
9	9	10	11	21	16	12	22	19	16

a. Construct a grouped frequency distribution with 6 classes.

b. What is the frequency of the 3rd class?

c. What is the frequency of the 6th class?

d. Determine the cumulative frequency distribution.

2. The following are scores of 80 students in a mathematics exam.

22	36	32	31	32	33	33	36	31	22
31	41	49	36	36	36	36	42	34	35
36	49	50	49	38	38	41	43	41	42
46	50	51	58	48	41	42	47	41	45
46	56	54	59	50	46	48	50	44	46
50	69	56	60	56	47	56	54	47	56
50	72	60	65	62	55	63	67	58	62
55	76	71	66	76	58	76	70	62	62

Construct a grouped frequency distribution, using 8 classes width. Answer the following questions.

a. Determine the number of classes.
b. Determine the lower-class limit of the fourth class.
c. Determine the upper-class limit of the fourth class.
d. Determine the frequency of the first class.
e. Determine the cumulative frequency distribution.
f. How many students got mathematics score less than 55?
g. Determine the cumulative frequency at the fifth class.

7.2.2 Grouped Continuous Data

Activity 7.4

Consider the following data.

46	47	60	63	69	49	50	60	60	70
50	50	50	63	63	48	48	49	70	70
49	49	60	61	62	70	51	51	70	70
55	60	45	46	46	60	55	57	59	60
67	50	55	55	69	47				

It represents weight of grade 11 students (in kg) in which the highest observed weight was 70 kg and the lowest was 45 kg. Discuss how this data can be organized into 5 categories using class width.

For Activity 7.4, you have to group this data in table form into 5 categories using class width that represents grouped frequency distribution that helps you to define grouped continuous data and other related definitions.

Definition 7.7

1. **Grouped continuous data** is a continuous data that has been grouped into categories.
2. **Class mark (Mid points)** is the average of the lower- and upper-class limits or the average of upper- and lower-class boundary. This means if m_i is the class mark (midpoint) of a class then it is calculated by

$$m_i = \frac{(\text{lower class boundary} + \text{upper class boundary})}{2}$$

Note

Steps for constructing grouped frequency distribution

1. Determine the number of classes required
2. Approximate the interval of each class or class width using the following formula

$$\text{Class width} = \frac{\text{Largest Value in ungrouped data} - \text{Smallest Value in ungrouped data}}{\text{number of classes required}}.$$

Example

On a certain construction site, the weekly wages (in Birr) of 100 labourers taken from a list (i.e., ungrouped data) in which the highest observed wage was Birr 258 and the lowest was Birr 142 are required to be given in 6 classes of a frequency distribution as follows. Let x be the weekly wage.

Table 7.4

Weekly wage (in Birr)	Number of labourers (*f*)
$140 \leq x < 160$	7
$160 \leq x < 180$	20
$180 \leq x < 200$	33
$200 \leq x < 220$	25
$220 \leq x < 240$	11
$240 \leq x < 260$	4
Total	**100**

Based on the above table, answer the following questions.

a. Construct a cumulative frequency distribution.

b. What is the range?

c. What is the class width?

d. What is the frequency of 4^{th} class?

e. How many people are paid Birr 220 and more?

Solution

a.

Table 7.5

Weekly wage (Birr) (Class interval)	Number of Laborers	Cumulative frequency (Cf)
$140 \leq x < 160$	7	7
$160 \leq x < 180$	20	27
$180 \leq x < 200$	33	60
$200 \leq x < 220$	25	85
$220 \leq x < 240$	11	96
$240 \leq x < 260$	4	100

b. Range = $258 - 142 = 116$.

c. Class width is 20.

d. The frequency of 4^{th} class is 25.

e. Number of people that are paid Birr 220 and more is 15.

Exercise 7.4

In a certain secondary school, the monthly wages (in Birr) of 100 teachers taken from a list (i.e. ungrouped data) in which the highest observed wage was Birr 11,000 and the lowest was Birr 4,200 are required to be given in 5 categories (classes) of a frequency distribution as shown in Table 7.6.

Table 7.6

Monthly wage (in Birr)	Number of Teachers(f)
$4{,}200 \leq x < 5{,}600$	23
$5{,}600 \leq x < 7{,}000$	16
$7{,}000 \leq x < 8{,}400$	30
$8{,}400 \leq x < 9{,}800$	20
$9{,}800 \leq x < 11{,}200$	11

Based on the above table, answer the following questions.

a. Construct a cumulative frequency distribution.

b. What is the range?

c. What is the frequency of 4^{th} class?

d. What is the class width?

e. Determine the lower-class limit for the third class.

f. What is the class mark of the second class?

g. How many people are paid Birr 7,000 and more?

7.3 Graphical Representation of Grouped Data

Activity 7.5

What knowledge do you have about graphical representation of grouped data such as histogram and frequency polygon?

In the previous section representation of data by tables has already been discussed. Now let us turn our attention to another representation of data, i.e., the graphical representation. Usually, comparisons among the group data are best shown by means of graphs. The representation then becomes easier to understand than the actual data. The histogram, and frequency polygon graph are most commonly applied graphical representation for continuous data.

Note

Steps for constructing histogram, and frequency polygon

1. Draw and label the x- and y- axis.
2. Choose a suitable scale for the frequencies or cumulative frequencies and label it on the y- axis.
3. Represent the class boundaries for the histogram or the mid points for the frequency polygon on the x -axis.
4. Plot the points.
5. Draw the bars or lines to connect the points.

Definition 7.8

Frequency Polygon is a graph constructed by using lines to join the midpoints of each interval, or bin. The frequency is placed along the vertical axis and classes mid points are placed along the horizontal axis.

Example

Construct histograms, and Frequency polygons for data shown in Table 7.7. Let x be the weekly wage.

Table 7.7

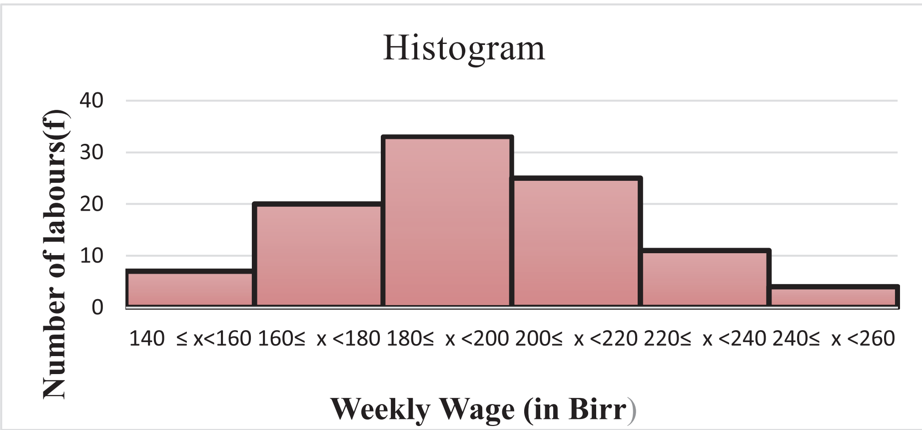
Weekly wage (in Birr)	**Number of labourers (*f*)**
$140 \leq x < 160$	7
$160 \leq x < 180$	20
$180 \leq x < 200$	33
$200 \leq x < 220$	25
$220 \leq x < 240$	11
$240 \leq x < 260$	4

Solution

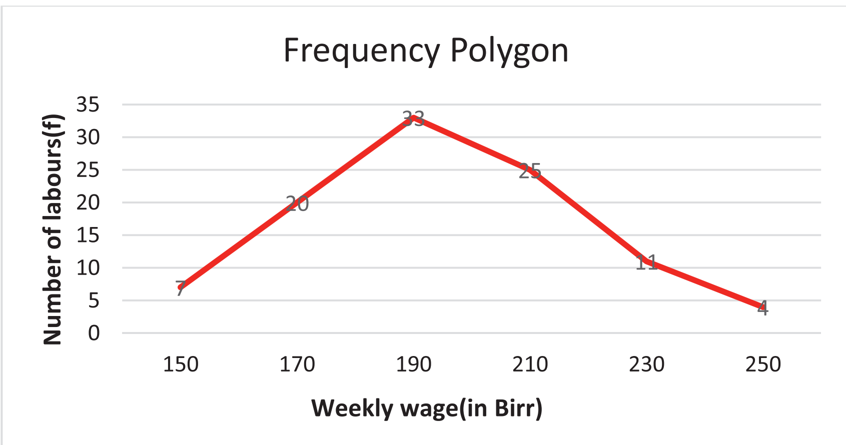
To construct histogram, and Frequency Polygon for the above data, you can apply the following steps;

1. Draw and label the x- and y- axis.
2. Choose a suitable scale for the frequencies and label it on the y- axis.
3. Represent the class limits for the histogram or the mid points for the frequency polygon on the x- axis.
4. Plot the points.
5. Draw the bars or lines to connect the points.

i) Histogram



ii) Frequency polygon



Exercise 7.5

1. Draw
   a. the histograms b. the Frequency polygons of the previous Exercise 7.4
2. The Table below shows weight of grade 11 section A male students.
   a. Draw the histogram b. Draw frequency polygon. Let the weight be w.

Table 7.8

Weight (w) (in kg)	No. of student
$40 \leq w < 45$	4
$45 \leq w < 50$	8
$50 \leq w < 55$	12
$55 \leq w < 60$	9
$60 \leq w < 65$	3

7.4 Measures of Central Tendency and Their Interpretation

Revision of mean for ungrouped data

Activity 7.6

1. Do you know how to calculate mean of ungrouped data?
2. Discuss how to calculate mean of the following ungrouped data.
   a. 8, 6, 8, 10 and 13
   b. 6, 7, 7, 5, 6, 2, 4 and 6

From Activity 7.6, you may recall how to calculate the mean for ungrouped data that you have learnt in grade 9.

Definition 7.9

The mean x of a set of data, denoted by $\overline{X}$ is equal to the sum of the data items divided by the number of items contained in the data set.

$$\overline{X} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} \quad \text{or}$$

If $x_1, x_2, \ldots, x_n$ is a set of data items, with frequencies $f_1, f_2, \ldots, f_n$ respectively, then their mean is given by

$$\overline{X} = \frac{f_1x_1 + f_2x_2 + \ldots + f_nx_n}{f_1 + f_2 + \ldots + f_n}$$

Example

Consider the following values which show the number of radios sold by an electronics shop for 26 days.

7	7	2	6	7	10	8	10	2	7
0	1	7	2	7	6	10	6	7	8
7	6	7	10	6	10				

a. Construct a frequency distribution table.

b. Find the mean from the raw data.

Solution

a. From the above data, you may have found the following frequency distribution table which shows the number of radios sold by the shop for 26 days;

Number of radios (x)	0	1	2	6	7	8	10
f	1	1	3	5	9	2	5

b. You use the above formula to find the mean from the frequency distribution table.

$$\overline{X} = \frac{f_1x_1 + f_2x_2 + f_3x_3 + f_4x_4 + f_5x_5 + f_6x_6 + f_7x_7}{f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7} = \frac{1 \times 0 + 1 \times 1 + 3 \times 2 + 5 \times 6 + 9 \times 7 + 2 \times 8 + 5 \times 10}{1 + 1 + 3 + 5 + 9 + 2 + 5}$$

$$= \frac{166}{26} = 6.38$$

Thus, the typical number of radios sold daily by the shop is about 6.4 or 62 % of the radios sold daily is above the average.

Exercise 7.6

1. Find the mean of the following data.

   3, 5, 7, 1, 8, 6

2. The following frequency distribution tables represent the score and weight of grade 11 students in mathematics. Find the mean for each of them.

   a.

Marks (out of 20)	5	9	10	12	13	19
f	2	10	5	5	7	1

   b.

Weight (kg)	41	43	52	54	60
f	1	14	15	21	14

7.4.1 Mean for Grouped Data

Activity 7.7

1. Do you know how to calculate mean of grouped data?
2. Suppose the first group of 8 students have mean score 7 in mathematics exam. A second and third group of 10 students and 12 students have mean score 8 and 6 in the same exam. What is the mean score of all students?

Form Activity 7.7, you have observed that the procedure for finding the mean for grouped data is similar to that of ungrouped data, except that the mid points of the classes are used for the x values.

Example 1

Calculate the mean of this grouped frequency table for students' test scores.

Table 7.9

Class interval	Frequency(f)
6 – 10	35
11 – 15	23
16 – 20	15
21 – 25	12
26 – 30	9
31 – 35	6

Solution

First, find the class marks (class midpoint).

Second, find the product of frequency class mark.

Third, find mean using the formula.

Table 7.10

Class interval	f_i	Midpoint (m_i)	$f_i m_i$
6 – 10	35	8	280
11 – 15	23	13	299
16 – 20	15	18	270
21 – 25	12	23	276
26 – 30	9	28	252
31 – 35	6	33	198
Total	100		1575

Now, total number of students = 100; Total marks(approximate) = 1575

$$\bar{x} = \frac{f_1m_1 + f_2m_2 + f_3m_3 + f_4m_4 + f_5m_5 + f_6m_6}{f_1 + f_2 + f_3 + f_4 + f_5 + f_6} = \frac{1575}{100} = 15.75$$

Hence, on average, the students' test score is approximately 15.75

Note

Steps to find the mean from a grouped frequency distribution

1. Find the class mark (midpoint) m_i of each class.
2. Multiply m_i by its corresponding frequency and add.
3. Divide the sum obtained in step 2 by the sum of the frequencies

$\bar{x} = \frac{f_1m_1 + f_2m_2 + f_3m_3 + \ldots + f_im_i}{f_1 + f_2 + f_3 + \ldots + f_i}$, where m_i is the class mark (midpoint) of a class and calculated by

$$m_i = \frac{\text{lower class boundary} + \text{upper class bundary}}{2}$$

Example 2

Table 7.11 indicates the age distribution of 30 students in a class of extension students attending evening classes. Find the mean age of these students.

Table 7.11

Age (in years)	Class midpoint (m_i)	Number of students(f_i)	$f_i\, m_i$
16 – 20	18	4	72
21 – 25	23	11	253
26 – 30	28	9	252
31 – 35	33	6	198
Total		30	775

Solution

$$\bar{x} = \frac{f_1m_1 + f_2m_2 + f_3m_3 + f_4m_4}{f_1 + f_2 + f_3 + f_4} = \frac{775}{30} = 25.83 \text{ years}$$

Thus, the average age of students in the class is about 25.83.

Example 3

Table 7.12 gives data on the height of 60 students in a class. Find the mean height of these students. Let the height be h.

Table 7.12

Height (in cm)	Class midpoint (m_i)	Number of students(f_i)	$f_i\,m_i$
$130 \leq h < 140$	135	5	675
$140 \leq h < 150$	145	6	870
$150 \leq h < 160$	155	18	2,790
$160 \leq h < 170$	165	22	3,630
$170 \leq h < 180$	175	9	1,575
Total		60	9540

Solution

$$\bar{x} = \frac{f_1m_1 + f_2m_2 + f_3m_3 + f_4m_4 + f_5m_5}{f_1 + f_2 + f_3 + f_4 + f_5} = \frac{9540}{60} = 159 \text{ cm.}$$

Thus, the average height of students in the class is about 159cm.

Exercise 7.7

1. The following frequency distribution tables represent the mid- year mark of grade 11 students in mathematics and their weight in kg. Find the mean for each of them.

a.

Marks (out of 50)	41−45	36−40	31−35	26−30	21−25	16−20
f	1	8	8	14	7	2

b.

Weight (kg)	f
$40 \leq w < 45$	1
$45 \leq w < 50$	**7**
$50 \leq w < 55$	8
$55 \leq w < 60$	**7**
$60 \leq w < 65$	**5**
$65 \leq w < 70$	5
$70 \leq w < 75$	**6**
$75 \leq w < 80$	1

2. The following table gives the frequency distribution of the number of people attacked by COVID-19 Pandemic per day for 50 consecutive days reported by Ministry of Health of Ethiopia. Calculate the mean.

No. of People attacked	10−12	13−15	16 −18	19 − 21	Total
f	4	12	20	14	50

Revision of Median for ungrouped Data

Activity 7.8

1. Do you know how to calculate the median of ungrouped data?
2. Discuss how to calculate median of the following ungrouped data.
   a. 9, 7, 9, 11 and 14
   b. 7, 8, 8, 6, 7, 3, 5 and 7

From Activity 7.8, you may have observed that median of ungrouped data is half way point in data, when the data is arranged in order. The median will be a value in the data or will fall between two values.

Note: Median is denoted by $(\tilde{X})$

Note

Steps in computing the median of ungrouped data

1. Arrange the data in increasing order.
2. Select the middle number.
3. 3. If n is odd, median = $\frac{(n+1)^{th}}{2}$ observation.
4. If n is even, median = $\frac{\left(\frac{n}{2}\right)^{th}\text{observation}+\left(\frac{n}{2}+1\right)^{th}\text{observation}}{2}$

Example 4

The following data shows the number of car accident for 9 days.

What will be the median of this data distribution?

7, 9, 6, 7, 11, 8, 4, 11, 7

Solution

Step 1. Arranging the data in an increasing order gives 4, 6, 7, 7, 7, 8, 9, 11,11

Step 2. Select the middle value

Since the number of observations is 9 and this number is odd; therefore,

$\widetilde{X} = \frac{(n+1)^{th}}{2}$ item

$= \left(\frac{9+1}{2}\right)^{th}$ item = 5^{th} item shows = 7

Thus, the median is 7.

Therefore, the median value of car accident is 7.

Example 5

Find the median from the following data; 70, 73, 69, 82, 60, 59.

Solution

Step 1: Arranging the data in an increasing order gives 59, 60, 69, 70, 73, 82.

Step 2: Select the middle value.

Since the number of observations is 6 and this number is even; therefore,

$$\tilde{X} = \frac{\left(\frac{n}{2}\right)^{th} \text{ item} + \left(\frac{n}{2}+1\right)^{th} \text{ item}}{2}.$$

$$= \frac{\left(\frac{6}{2}\right)^{th} \text{ item} + \left(\frac{6}{2}+1\right)^{th} \text{ item}}{2}.$$

$$= \frac{3^{rd} \text{ item} + 4^{th} \text{ item}}{2}.$$

$$= \frac{69+70}{2} = \frac{139}{2} = 69.5$$

Hence, the median is 69.5. That is, the median value of the data is 69.5.

Exercise 7.8

1. Consider the following data which shows grade 11 students mathematics score out of 20.

10	11	12	11	13	15	15	13	12	11
10	9	13	12	11	10	9	13	13	12
11	10	11	12	13	15	13	12	11	10
20	20								

   Find the median from the raw data.

2. The Bills paid (in Birr) for water consumption by Ato Girma in the last 8 months is as follows.

   62, 78, 57, 66, 88, 58, 113, 92.

   a. Find the median of Bills paid for the water consumption.
   b. Calculate the mean and compare it with the median.

7.4.2 The Median of Grouped Data

Activity 7.9

1. Do you know how to calculate the median of grouped data?
2. Discuss how to calculate the median of the following data on mathematics test score of 20 students out of 10 (full mark).

Math test Score	1 – 2	3 – 4	5 – 6	7 – 8	9 – 10
f	2	4	8	5	1

From Activity 7.9, you can observe that the procedure for finding the median from grouped data is similar to that of ungrouped data; except preparing the cumulative frequency distribution.

Note

Steps to find the median of a grouped frequency distribution

1. Prepare a cumulative frequency distribution.
2. Find the class where the median is located. It is the lowest class for which the cumulative frequency equals or exceeds $\frac{n}{2}$.
3. Determine the median by the formula

$$\tilde{X} = B_L + \left(\frac{\frac{n}{2} - cf_B}{fc}\right)i$$

where,

B_L = Lower limit of the class containing the median (median class)

n = total number of observations.

cf_B = the cumulative frequency in the class preceding ("coming before") the class containing the median.

fc = the number of observations (frequency) in the class containing the median.

i = the size of the class interval. (i.e. width of the median class)

Example

The following is the time 50 students take to travel to school. Find the median time to travel to school. Let t be the time.

Time travel to school(min)	$1 \le t < 10$	$10 \le t < 20$	$20 \le t < 30$	$30 \le t < 40$	$40 \le t < 50$
f	8	14	12	9	7

Solution

From this, you can prepare the cumulative frequency column as follows;

Table 7.12

Time to travel to school(min)	**f**	**Cumulative frequency (Cf)**
$1 \le t < 10$	8	8
$10 \le t < 20$	14	8 + 14 = 22
$20 \le t < 30$	**12**	**22 + 12=34**
$30 \le t < 40$	9	34 + 9 = 43
$40 \le t < 50$	7	43 + 7 =50
Total	50	

The median class is the class containing the $\left(\frac{50}{2}\right)^{th}$ item = 25^{th} item. In this case, it is in the 3^{rd} class.

Therefore, the median class is **$20 \le t < 30$**

Thus, $B_L = 20$, $\frac{n}{2} = 25$, $fc = 12$, i = 40 – 30 = 10, $cf_B = 22$

Therefore,

$$\tilde{X} = B_L + \left(\frac{\frac{n}{2} - cf_B}{fc}\right)i$$

$$= 20 + \left(\frac{25-22}{12}\right)10 = 20 + 2.5 \quad = 22.5$$

Thus, the median time to travel to school is 22.5 minutes.

Exercise 7.9

1. The following data shows the number of people attacked by COVID-19 epidemic within 40 days.

57	59	54	57	58	56	59	53	59	57
53	54	56	53	54	57	54	56	58	55
56	53	55	52	54	53	54	57	58	55
58	56	57	60	56	57	61	61	57	56

a. Find the median from the raw data.

b. Construct a grouped frequency distribution with 5 classes.

2. Calculate the median of the following data about weight of students in a class. Let w be the weight.

Weight (in kg)	Number of students
$35 \leq w < 40$	8
$40 \leq w < 45$	9
$45 \leq w < 50$	14
$50 \leq w < 55$	15
$55 \leq w < 60$	10
$60 \leq w < 65$	4

3. The following set of data shows about mark of students in mathematics exam:

Marks	41–47	48–54	55–61	62–68	69–75	76–82	83–89	90–96
No of students	4	7	9	14	9	8	5	4

Find the median marks of the students.

7.4.3 Quartiles, Deciles and Percentiles

Measures of central tendency like median is a measure of location that divides the data into two equal parts. In addition, there are other measures of position or location such as percentiles, deciles, and quartiles. These measures are used to locate the relative position of a data value in the data set. For example, if a value is located at the 80th percentile, it means that 80% of the values fall below it in the distribution and 20% of the values fall above it. The *median* is the value that corresponds to the 50th percentile, since one-half of the values fall below it and the remaining half of the values fall above it. Before discussing these measures of position for grouped data, let us see how they work for ungrouped data and then move to grouped data.

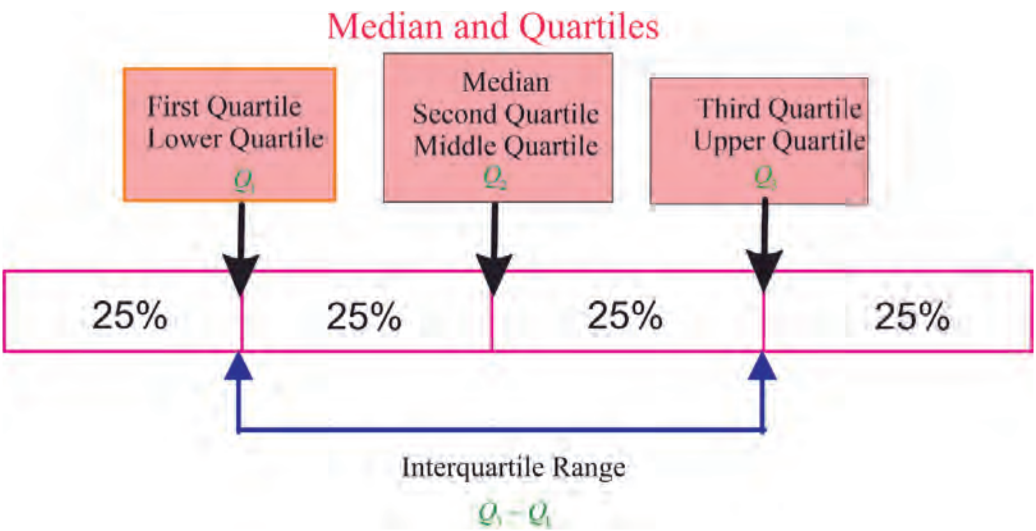
7.4.3.1 Quartiles, Deciles and Percentiles for Ungrouped Data

1. Quartiles

Activity 7.10

1. Do you know how to calculate the quartiles of ungrouped data?
2. Explain meaning and steps to calculate quartiles for ungrouped data?

A quartile divides data into three points, namely **a lower quartile, median**, and **upper quartile** to form four groups of the dataset. The lower quartile, or first quartile, is denoted as Q_1 and is the middle number that falls between the smallest value of the dataset and the median. The second quartile, Q_2, is also the median. The upper or third quartile, denoted as Q_3, is the central point that lies between the median and the highest number of the distribution.



Now, the four groups formed from the quartiles are the first group of values contains **the smallest number up to Q_1**; the second group includes **Q_1 to the median**; the third set is **the median to Q_3;** the fourth category comprises **Q_3 to the highest data point of the entire** set.

Definition 7.10

The difference of the upper quartile and the lower quartile is **the interquartile range (IQR); i.e., IQR = $Q_3 - Q_1$**

Example 1

Suppose the distribution of math scores in a class of 20 students in ascending order is:

59	60	65	65	68	69	70	72	75	75
76	77	81	82	84	87	90	93	95	98

Then, find Q_1, Q_2 and Q_3 and interpret what the value of each shows?

Solution

First, mark down the median, Q_2, which in this case is the average of the 10^{th} and 11^{th} score: 75.5.

Q_1 is the central point between the smallest score and the median. In this case, Q_1 lies between the 5^{th} and 6^{th} score: 68.5 (Note that the median can also be included when calculating Q_1 or Q_3 for an even score). If you were to include the median on either side of the middle point, then Q_1 will be the middle value between the 5^{th} and 6^{th} scores. It is the average of the 5^{th} and 6^{th} scores.

i.e., $\frac{(5^{th} + 6^{th})}{2} = \frac{(68 + 69)}{2} = 68.5.$

Q_3 is the middle value between Q_2 and the highest score: 98. $Q_3 = \frac{(84 + 87)}{2} = 85.5$

This means, Q_1 tells us that 25% of the scores are less than 68.5 and 75 % of the class scores are greater than 68.5. Q_2 (the median) shows that 50% of the scores are less than 75.5, and 50% of the scores are above 75.5. Finally, Q_3 reveals that 25% of the scores are greater than 85.5 and 75% are less than 85.5.

Note

Steps to calculate quartiles for ungrouped data

1. Arrange the data in increasing order of magnitude.
2. If the number of observations is:

   i) odd, $Q_k = \left(\frac{k(n+1)}{4}\right)^{th}$ item

   ii) Even, $Q_k = \left(\frac{\left(\frac{kn}{4}\right)+\left(\frac{kn}{4}+1\right)}{2}\right)^{th}$ item

Example 2

Find Q_1, Q_2, Q_3 and IQR for the following data;

30, 43, 47, 51, 36, 34, 26, 14, 11, 5, 7, 54, 57

Solution

Arranging the data in increasing order of magnitude, you get,

5, 7, 11, 14, 26, 30, 34, 36,43, 47, 51, 54, 57

$Q_1 = \left(\frac{1(13+1)}{4}\right) = (3.5)^{th}$ item. What does this mean?

Q_1 lies between the 3^{rd} and 4^{th} items.

Therefore, $Q_1 = 3^{rd}$ item + $0.5(4^{th}$ item – 3^{rd} item)

$= x_3 + 0.5\ (x_4 - x_3)$

$= 11 + 0.5(14-11) = 11 + 1.5 = 12.5$

Thus, $Q_1 = 12.5$.

$Q_2 = \left(\frac{2(13+1)}{4}\right) = 7^{th}$ item

Thus, Q_2 is the 7^{th} item

Therefore, $Q_2 = 34$.

$Q_3 = \left(\frac{3(13+1)}{4}\right) = 10.5^{th}$ item

It is half of the way between the 10^{th} (x_{10}) and 11^{th} (x_{11}) items

Thus, $Q_3 = x_{11} + 0.5\,(x_{11} - x_{10})$

$= 47 + 0.5(47 - 43) = 47 + 2 = 49$

$IQR = Q_3 - Q_1 = 49 - 12.5 = 36.5.$

Exercise 7.10

1. For each of the following data sets:

   a. 12, 5, 22, 30, 7, 36, 14, 42, 15, 53, 25, 58, 62,63, 55

   b. 10, 13, 15, 12, 18, 15, 16, 12, 13, 20, 17, 14, 19, 18, 23,28

   c.

Score	15	19	20	22	24	25	31
f	11	16	15	7	5	5	2

   Find Q_1, Q_2, and Q_3

2. The marks obtained by 19 students of a class are given below:

   27, 36, 22, 31, 25, 26, 33, 24, 37, 32, 29, 28, 36, 35, 27, 26, 32, 35 and 28.

   Find:

   a. median b. lower quartile c. upper quartile d. interquartile range

2. Deciles

Activity 7.11

1. Do you know how to calculate the deciles of ungrouped data?
2. Explain meaning and steps to calculate deciles for ungrouped data?

Deciles divide a set of data into ten equal number of parts. There are nine deciles. Like the quartile, a decile consists of nine data points that divide a data set into 10 equal parts and it is usually used to assign decile ranks to a data set. A decile rank arranges the data in order from lowest to highest and is done on a scale of one to 10 where each successive number corresponds to an increase of 10 percentage points. In other words, there are nine decile points. Namely, D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8, and D_9. The 1st decile, or D_1, is the point that has 10% of the observations lies below it, D_2 has 20% of the observations lies below it, D_3 has 30% of the observations lies below it, and so on. The example below clarifies the point.

Example 3

The table below shows the ungrouped scores (out of 100) for 29 exam takers:

45	52	55	57	58	60	61	64	65	66
69	72	73	75	76	78	81	82	84	87
88	90	91	92	93	94	95	96	97	

Using the information presented in the table, find D_1, D_3 and D_5 and interpret what the value of each shows.

Solution

Using the information presented in the table, the 1st decile can be calculated as:

$$D_1 = \text{Value of } \left(\frac{29+1}{10}\right)^{th} \text{ data}$$

$$= \text{Value of } 3^{rd} \text{ data}$$

$$= 55$$

Thus, D_1 means that 10% of the scores of exam takers falls below 55.

Let's calculate the 3rd decile:

$$D_3 = \frac{\text{value of } 3(29\ +1)^{th}}{10}$$

$$= \text{Value of } 9^{th} \text{ position.}$$

$$= 65$$

This means, 30% of the scores of exam takers lies below 65.

What would you get if we were to calculate the 5th decile?

$$D_5 = \frac{\text{value of } 5(29\ +1)^{\text{th}}}{10}$$

$$= \text{Value of } 15^{\text{th}} \text{ position}$$

$$= 76$$

This means, 50% of the scores of exam takers fall below 76.

The 5^{th} decile is also the median of the scores of exam takers. Looking at the data in the table, the median can be calculated as 76 = median = D_5. At this point, half of the scores of exam takers lie above and below the distribution.

In general, follow these steps, to calculate deciles;

Note

Steps to calculate deciles for ungrouped data

1. Arrange the data in increasing order of magnitude.
2. If the number of observations is:

i) odd, $D_j = \left(\frac{j(n+1)}{10}\right)^{th}$ item. ii) even, $D_j = \left(\frac{\left(\frac{jn}{10}\right) + \left(\frac{jn}{10} + 1\right)}{2}\right)^{th}$ item.

Example 4

Find D_3 and D_8 for the following data: 27, 40, 44, 48, 52, 33, 31, 23, 11, 7.

Solution

Arranging in increasing order of magnitude; you get,

7, 11, 23, 27, 31, 33, 40, 44, 48, 52.

$$D_3 = \left(\frac{\left(\frac{3(10)}{10}\right) + \left(\frac{3(10)}{10} + 1\right)}{2}\right)^{th} \text{ item}$$

$$= \left(\frac{3+4}{2}\right)^{th} \text{ item} = (3.5)^{th} \text{ item} = \frac{(23+27)}{2} = 25$$

This means, 30% of the data lies below 25.

$$D_8 = \left(\frac{\left(\frac{8(10)}{10}\right) + \left(\frac{8(10)}{10} + 1\right)}{2}\right)^{th} \text{ item}$$

$$= \left(\frac{8+9}{2}\right)^{th} \text{ item} = (8.5)^{th} \text{ item} = \frac{(44+48)}{2} = 46$$

Thus, 80% of the data lies below 46.

Exercise 7.11

Find D_2, D_3, D_5, D_8 for each of the following data sets:

a. 12, 5, 22, 30, 7, 36, 14, 42, 15, 53, 25, 58, 62,63, 55

b. 10, 13, 15, 12, 18, 15, 16, 12, 13, 20, 17, 14, 19, 18, 23, 2

c.

Score	15	19	20	22	24	25	31
f	11	16	15	7	5	5	2

3. Percentiles

Activity 7.12

1. Do you know how to calculate the Percentiles of ungrouped data?
2. Explain meaning and steps to calculate Percentiles for ungrouped data?

Percentiles divide the data set into 100 equal groups. There are 99 percentiles, namely, P_1, P_2, … P_{99}.

Percentiles are also used to compare an individual's test score with the national norm. For example, tests such as the National examination are taken by students in 8^{th} grade. A student's scores are compared with those of other students locally and nationally by using percentile ranks.

Percentiles and percentages are two different concepts. Thus, the percentage is a mathematical quantity which is written out of a total of 100 whereas a percentile is the percentage of values found under the specific values. Percentiles are mostly used for the ranking system. For instance, if you get 72 correct answers out of a possible 100. You obtain a percentage score of 72. There is no indication of your position with respect to the rest of the class. You could have scored the highest, the lowest, or somewhere in between. On the other hand, if a raw score of 72 corresponds to the 64^{th} percentile, then you did better than 64% of the students in your class.

Note

Steps to calculate Percentiles for ungrouped data

1. Arrange the data in increasing order of magnitude.
2. If the number of observations is:

   i) odd, $P_j = \left(\frac{j\,(n+1)}{100}\right)^{th}$ item. ii) even, $P_j = \left(\frac{\left(\frac{jn}{100}\right) + \left(\frac{jn}{100} + 1\right)}{2}\right)^{th}$ item.

Example 5

Find P_{47} and P_{83} for the following data:

27	40	57	48	52	33	31	23	11	7
35	34	24	34	59	35	10	23	54	18
29	43	45	47	49	37	42	34	45	47
23	39	55	37	58	48	26	37	17	49

Solution

Arranging in increasing order of magnitude; you get,

7,10, 11, 18, 23, 23, 23, 24, 26, 27, 29, 31, 33, 34, 34, 34, 35, 35, 37, 37, 37, 39, 40, 42, 43, 45, 45, 47, 47, 48, 48, 49, 49, 53, 54, 55, 57, 58, and 59.

$$P_{47} = \left(\frac{\left(\frac{47(40)}{100}\right) + \left(\frac{47(40)}{100} + 1\right)}{2}\right)^{th} \text{ item}$$

$$= \left(\frac{18.8+19.8}{2}\right)^{th} \text{ item} = (19.3)^{\text{th}} \text{ item} = 37 + 0.3(37 - 37) = 37$$

$$P_{83} = \left(\frac{\left(\frac{83(40)}{100}\right) + \left(\frac{83(40)}{100} + 1\right)}{2}\right)^{th} \text{ item}$$

$$= \left(\frac{33.2 + 34.2}{2}\right)^{\text{th}} \text{ item} = (33.7)^{\text{th}} \text{ item} = 49 + 0.7(53 - 49) = 49 + 0.7(4)$$

$$= 51.8$$

Note that $P_{83} = 51.8$, that means 83% of the data values are less than 51.8 and the rest are above it.

Exercise 7.12

Find P_{50}, P_{24}, and P_{87} for each of the following data sets:

a. 12, 5, 22, 30, 7, 36, 14, 42, 15, 53, 25, 58, 62,63, 55

b. 10, 13, 15, 12, 18, 15, 16, 12, 13, 20, 17, 14, 19, 18, 23, 2

c.

Score	15	19	20	22	24	25	31
f	11	16	15	7	5	5	2

7.4.3.2 Quartiles, Deciles and Percentiles for Grouped data

When you have a very large raw data, grouping the data in a frequency distribution will make it easier.

1. Quartiles for grouped data

Activity 7.13

1. Do you know how to calculate the Quartiles for grouped data?
2. Explain steps to calculate quartiles for grouped data by giving your own examples.

Just like quartiles for ungrouped data, when the observation is arranged in increasing order then the values that divide the whole data into four (4) equal parts are called quartiles for grouped frequency distributions. These values are denoted by Q_1, Q_2 and Q_3. It is to be noted that 25% of the data falls below Q_1, 50% of the data falls below Q_2 and 75% of the data falls below Q_3.

Example 6

Find the quartiles of the following grouped data

Score	41 – 50	51– 60	61– 70	71– 80	81– 90	91– 100	101– 110
f	17	29	20	12	9	25	8

Solution

Table 7.13

Score	f	cf
41 – 50	17	17
51 – 60	29	46
61 – 70	20	66
71 – 80	12	78
81 – 90	9	87
91 – 100	25	112
101 – 110	8	120

Q_1 is the 30^{th}item in the distribution. By assuming that the items are equally spread through each class, we calculate the value of the required item by means of proportions.

Now, since the first 17 items lie in earlier classes, Q_1 is the $30 - 17 = 13^{th}$item in a class of 29 items. This means it lies $\left(\frac{13}{29}\right)^{\text{th}}$ of the way into the class. Since this class has an interval length of 10, this means that $\frac{13}{29} \times 10 = 4.48$ is to be added to the lower end. Now, the quartile class starts at 51, so that the first quartile is $51 + 4.48 = 55.48$.

Similarly, $Q_3 = 91 + \frac{90-87}{25} \times 10 = 91 + 1.2 = 92.2$.

Thus, let us summarize the above example in the following formula:

The k^{th} **quartile** for a grouped frequency distribution is:

$$Q_K \ (k^{th} \text{ quartile}) = B_L + \left(\frac{\frac{kn}{4} - cf_B}{fc}\right)i$$

$k = 1, 2, 3$ and

B_L = lower class limit of the k^{th} quartile class

n = number of observations

cf_B = the cumulative frequency before the k^{th} quartile class

f_c = the number of observations (frequency) in the k^{th} quartile class

i = the size of the class interval

Note

Steps to find quartiles for grouped data

1. Prepare a cumulative frequency distribution.
2. Find the class where the k^{th} quartile belongs: the $\left(\frac{kn}{4}\right)^{th}$ item.
3. Use the formula above.

Example 7

Find Q_1, Q_2, Q_3 and IQR of the following distribution;

Table 7.14

Score	f	cf
141 – 150	21	21
151 – 160	19	40
161 – 170	15	55
171 – 180	17	72
181 – 190	8	80
191 – 200	5	85
201 – 210	4	89
211 – 220	3	92
221 – 230	2	94
231 – 240	2	96
241 – 250	4	100

Solution

$n = 100$

Q_1 is $\left(\frac{100}{4}\right)^{th}$ item; that is, 25^{th} item which falls in the 2^{nd} class. $cf_B = 21$, $f_c = 19$ and $i = 10$

$$Q_1 = B_L + \left(\frac{\frac{1n}{4} - cf_B}{fc}\right)i$$

$$= 151 + \left(\frac{\frac{1 \times 100}{4} - 21}{19}\right)10$$

$= 151 + \left(\frac{25-21}{19}\right)10$

$= 151 + \left(\frac{4}{19}\right)10$

$= 151 + 2.11$

$= 153.11$

Q_2 is $\left(\frac{2 \times 100}{4}\right)^{th}$ item; that is, 50^{th} item, it is found in the 3^{nd} class. $cf_B = 40$, $f_c = 15$ and $i = 10$

$Q_2 = B_L + \left(\frac{\frac{2n}{4} - cf_B}{fc}\right)i$

$= 161 + \left(\frac{\frac{2 \times 100}{4} - 40}{15}\right)10$

$= 161 + \left(\frac{50-40}{15}\right)10$

$= 161 + \left(\frac{10}{15}\right)10 = 161 + 6.7 = 167.7$

Q_3 is $\left(\frac{3 \times 100}{4}\right)^{th}$ item; that is, 75^{th} item, and Q_3 found in the 5^{th} class. $cf_B = 72$, $f_c = 8$ and $i = 10$

$Q_3 = B_L + \left(\frac{\frac{3 \times n}{4} - cf_B}{fc}\right)i$

$= 181 + \left(\frac{\frac{3 \times 100}{4} - 72}{8}\right)10$

$= 181 + \left(\frac{75-72}{8}\right)10$

$= 181 + \left(\frac{3}{8}\right)10 = 181 + 3.7 = 184.7$

$IQR = Q_3 - Q_1 = 184.7 - 153.11 = 31.59$

Note

Q_2 = median i.e., the 2^{nd} quartile is the same as the median.

Exercise 7.13

1. The following table gives the amount of time (in minutes) spent on the internet each evening by a group of 56 students. Let t be the time.

Time spent on internet	$10 \le t < 15$	$15 \le t < 20$	$20 \le t < 25$	$25 \le t < 30$	$30 \le t < 35$
No of students	9	10	18	10	9

Find Q_1, Q_2, and Q_3

2. Find Q_1, Q_2 and Q_3 of the following data. It is a distribution of marks obtained in a mathematics exam (out of 50).

Score	30–32	33–35	36–38	39–41	42–44	45–47
Number of students	12	18	16	12	8	6

   a. From the above data, if students in the top 25% are to be awarded a certificate, what is the minimum mark for a certificate?

   b. If students whose scores are in the bottom 25% of the marks are considered as failures, then what is the maximum failing mark?

2. Deciles for grouped data

Activity 7.14

1. Do you know how to calculate the deciles for grouped data?
2. Explain steps and formulas to calculate deciles for grouped data by giving your own examples.

Just like deciles for ungrouped data, when the observation is arranged in increasing order, then the values that divide the whole data into ten (10) equal parts are called deciles for grouped frequency distributions. These values are denoted by D_1, D_2, ...,

D_9. It is to be noted that 10% of the data falls below D_1, 20% of the data falls below D_2, ..., and 90% of the data falls below D_9.

The **j^{th} decile** for grouped frequency distributions is calculated in a similar way as follows;

$$D_j(jth\ decile) = B_L + \left(\frac{\frac{jn}{10} - cf_B}{fc}\right)i$$

$j = 1, 2, 3, \ldots, 9$ and

B_L = lower class limit of the j^{th} decile class.

n = number of observations.

cf_B = the cumulative frequency before the j^{th} decile class.

f_c = the number of observations (frequency) in the j^{th} decile class, and

i = the size of the class interval

Note

Steps to find deciles for grouped data

1. Prepare a cumulative frequency distribution.
2. Find the class where the j^{th} decile belongs: the $\left(\frac{jn}{10}\right)^{th}$ item
3. Use the formula above

Example 8

Find D_6 and D_8 of the following data.

Table 7.15

Score	f	cf
141 – 150	21	21
151 –160	19	40
161 – 170	15	55
171 – 180	17	72
181 – 190	8	80
191 – 200	5	85
201 – 210	4	89
211 – 220	3	92
221 – 230	2	94
231 – 240	2	96
241 – 250	4	100

Solution

i) D_6 is $\left(\frac{6 \times 100}{10}\right)^{th}$ item = $(60)^{th}$ item, it is found in the 4th class. $cf_B = 55$, $f_c = 17$ and

i = 10

Thus,

$$D_6 = B_L + \left(\frac{\frac{6n}{10} - cf_B}{fc}\right)i$$

$$= 171 + \left(\frac{\frac{6\times100}{10} - 55}{17}\right)10$$

$$= 171 + \left(\frac{60 - 55}{17}\right)10$$

$$= 171 + \left(\frac{5}{17}\right)10 = 171 + 2.94 = 173.94$$

ii) D_8 is $\left(\frac{8 \times 100}{10}\right)^{th}$ item = $(80)^{th}$ item, it is found in the 5th class. $cf_B = 72$, $f_c = 8$

and i = 10

$$\text{Thus, } D_8 = B_L + \left(\frac{\frac{8n}{10} - cf_B}{fc}\right)i$$

$$= 181 + \left(\frac{\frac{8\times100}{10} - 72}{8}\right)10$$

$$= 181 + \left(\frac{80 - 72}{8}\right)10$$

$$= 181 + \left(\frac{8}{8}\right)10 = 181 + 10 = 191$$

Exercise 7.14

1. The following Table gives the amount of time (in minutes) spent on the internet each evening by a group of 41 students. Let t be the time.

   Find D_3, D_5, and D_7.

Time spent on internet	$10 \le t < 15$	$15 \le t < 20$	$20 \le t < 25$	$25 \le t < 30$	$30 \le t < 35$
No of students	3	9	10	12	7

2. The following Table shows the frequency distribution of monthly income of workers in a factory. Let x be the income in thousands.

Find, D_2, D_5 and D_8

Income (in thousands)	Number of workers
$1 \le x < 5$	10
$5 \le x < 9$	12
$9 \le x < 13$	8
$13 \le x < 17$	7
$17 \le x < 21$	5
$21 \le x < 25$	8
$25 \le x < 29$	4
$29 \le x < 33$	6

3. Percentiles for grouped data

Activity 7.15

1. Do you know how to calculate the percentiles for grouped data?
2. Explain steps and formulas to calculate percentiles for grouped data by giving your own examples.

Just like percentiles for grouped data, when the observation is arranged in increasing order then the values that divide the whole data into hundred (100) equal parts are called percentiles for grouped frequency distribution. These values are denoted by P_1, P_2, ..., P_{99}. It is to be noted that 1% of the data falls below P_1, 2% of the data falls below P_2, ..., and 99% of the data falls below P_{99}.

$$P_j(jth \text{ percentile}) = B_L + \left(\frac{\frac{jn}{100} - cf_B}{fc}\right)i$$

Where $j = 1, 2, 3, \ldots, 99$ and

B_L = lower class limit of the j^{th} percentile class;

n = number of observations;

cf_B = the cumulative frequency before the j^{th} percentile class;

f_c = the number of observations (frequency) in the j^{th} *percentile* class; and

i = the size of the class interval.

Note

Steps to find percentile for grouped data

1. Prepare a cumulative frequency distribution.
2. Find the class where the jth percentile belongs: the $\left(\frac{jn}{100}\right)^{th}$ item
3. Use the formula $P_j(jth \text{ percentile}) = B_L + \left(\frac{\frac{jn}{100} - cf_B}{fc}\right)i$

Example 9

Find $P_{35,}$ and P_{91} of the following data.

Table 7.16

Score	f	Cumulative frequency (Cf)
141 – 150	21	21
151 – 160	19	40
161 – 170	15	55
171 – 180	17	72
181 – 190	8	80
191 – 200	5	85
201 – 210	4	89
211 – 220	3	92
221 – 230	2	94
231 – 240	2	96
241 – 250	4	100

Solution

i) P_{35} is $\left(\frac{35 \times 100}{100}\right)^{th}$ item $= (35)^{th}$ item, and it is found in the 2nd class.

$cf_B = 21$, $f_c = 19$ and $i = 10$

Thus, $p_{35} = B_L + \left(\frac{\frac{35n}{100} - cf_B}{fc}\right)i$

$$= 151 + \left(\frac{\frac{35 \times 100}{100} - 21}{19}\right)10 = 151 + \left(\frac{35 - 21}{19}\right)10 = 151 + \left(\frac{14}{19}\right)10$$

$$= 151 + 7.37 = 158.37$$

This means, 35% of the data falls below 158.37.

ii). P_{91} is $\left(\frac{91\times100}{100}\right)^{th}$ item = $(91)^{th}$ item, it is found in the 8^{th} class. $cf_B = 89$, $f_c = 3$ and $i = 10$

Thus, $p_{91} = B_L + \left(\frac{\frac{91n}{100} - cf_B}{fc}\right)i$

$$= 211 + \left(\frac{\frac{91\times100}{100} - 89}{3}\right)10 = 211 + \left(\frac{91-89}{3}\right)10$$

$$= 211 + \left(\frac{2}{3}\right)10$$

$$= 217.67$$

This means, 91% of the data falls below 217.67.

Exercise 7.15

1. The following Table shows mathematics test scores of 100 students.

   Find P_{25}, P_{61}, P_{80}

Scores	21 – 30	31– 40	41 –50	51 – 60	61–70	71 – 80	81– 90	91– 100
No of Students	4	18	9	13	14	10	24	8

2. The following Table gives the amount of time spent (in minutes) spend studying for a mathematics test by a group of 41 students. Let t be the time.

   Find P_{33}, P_{50} and P_{75}.

Time spent on studying(min)	$30 \le t < 35$	$35 \le t < 40$	$40 \le t < 45$	$45 \le t < 50$	$50 \le t < 55$
No of students	3	12	20	2	4

3. The following Table gives a frequency distribution of weight (in kilogram) of 57 children at a day care centre. Let w be the weight.

Weight(kg)	No children
$10 \le w < 20$	5
$20 \le w < 30$	19
$30 \le w < 40$	10
$40 \le w < 50$	13
$50 \le w < 60$	4
$60 \le w < 70$	4
$70 \le w < 80$	2

Calculate

a. The maximum weight of lower 30% of the children

b. The minimum weight of upper 30% of the children.

7.4.4 The Mode for a Grouped Data

Activity 7.16

1. Do you know how to calculate the mode of ungrouped data as you have seen in grade 9?
2. Consider the following sets of data and discuss the most frequently occurring value in the data.

   a. 5, 3, 5, 8 and 9 b. 8, 9, 9, 7, 8, 2 and 5

From Activity 7.16, you have observed the most frequently occurring value for a data.

Definition 7.11

The mode is the value that occurs most often in a data. Mode is denoted by $(\hat{X})$

A data set that has only one value that occurs with the greatest frequency is said to be **unimodal.**

If a data set has two values that occur with the same greatest frequency, both values are considered to be the mode and the data set is said to be **bimodal.**

If a data set has more than two values that occur with the same greatest frequency, each value is used as the mode, and the data set is said to be **multimodal.**

When no data value occurs more than once, the data set is said to have ***no mode***. A data set can have more than one mode or no mode at all. These situations will be shown in some of the examples that follow.

Example 1

Find the mode of the bonuses of students for one semester. The bonuses are:

a. 18, 14, 15, 10, 11, 3, 10, 12, 10 marks.

b. 6, 10, 9, 6, 10, 4 marks.

c. 5, 6, 7, 11, 12 marks.

d.

x	7	12	15	17	19	23
f	6	4	6	5	6	5

Solution

a. In this observation, the most frequent value is 10. Therefore, the mode is $\widehat{X} = 10$ since it appears three times. This data has only one mode and is called unimodal.

b. Here both 6 and 10 appear twice but the rest appear only once. Hence the modes are 6 and 10. This distribution has two modes. Such distributions are said to be **bimodal**.

c. Every number appeared only once. Hence there is no mode for this distribution.

d. Three values 7, 15 and 19 all appear 6 times. Hence the modes are 7, 15 and 19. Distributions that have more than two modes are called **multimodal.**

Activity 7.17

1. Do you know how to calculate the mode of grouped data?
2. Explain steps to find mode from grouped data and calculate mode of grouped data by giving your own examples.

For Activity 7.17, just like ungrouped data, you have to see the frequency of each number or variable, and the variable that has the greater frequency is the mode. This changes when you work with grouped data, because when you work with grouped data there are no numbers to count how many times each number is repeated. Instead the data is organized in intervals and the way the mode is found changes.

Now, before you find any mode(s) that might exist, check the class interval of all classes should be equal (uniform class interval).

You can use the following steps to compute the mode of grouped or continuous frequency distribution with equal class intervals:

Step 1: Prepare the frequency distribution table in such a way that its first column consists of the observations and the second column the respective frequency.

Step 2: Determine the class of maximum frequency by inspection. This class is called the modal class.

Step 3: To calculate mode, use the following formula:

$$\text{Mode} = \widehat{X} = B_L + \left(\frac{d_1}{d_1 + d_2}\right)i$$

Where,

B_L = lower class limit of the modal class.

d_1= the difference between the frequency of the modal class(f_{mo}) and the frequency of the preceding class (pre-modal class) = (f_{mo}) – f_1

d_2= the difference between the frequency of the modal class and the frequency of the subsequent class (next class) = (f_{mo}) – f_2

i = size of the class interval.

Example 2

Following is the distribution of the lifespans, in days, of the bees in a colony. Calculate the mode of the distribution. **Table 7.17**

Lifespan(days)	Number of bees
16 – 35	10
36 – 55	12
56 – 75	19
76 – 95	31
96 – 115	33
116 – 135	7
136 – 155	5

Solution

96 – 115 is the modal class since it is a class with the highest frequency.

$B_L = 96$ $\quad d_1 = 33 - 31 = 2$ $\quad d_2 = 33 - 7 = 26$ $\quad i = 115 - 96 = 20$

$$\hat{X} = B_L + \left(\frac{d_1}{d_1 + d_2}\right)i$$

$$= 96 + \left(\frac{2}{28}\right)20 = 96 + 1.36 = 97.43$$

Exercise 7.16

1. The following is a distribution of the number of drivers who violated traffic-safety in Addis Ababa. Find mode of the distribution.

Violated traffic safety	1–3	4–7	8–11	12–15	16 –19	20 – 23
No of drivers	6	10	20	22	6	2

2. The following represents the number of minutes that students spent studying for a math test. Find the mode of distribution. Let t be the time.

Studying time (in minutes)	$0 \leq t < 10$	$10 \leq t < 20$	$20 \leq t < 30$	$30 \leq t < 40$	$40 \leq t < 50$
No of students	2	10	6	4	3

7.5 Real-life Application of Statistics

There are numerous applications of statistics in different professions. For instance, statistics can be applied in clinical trial and design, corporate sectors, weather forecasting, sports and financial markets. Moreover, it can be used in a variety of domains including business, industry, agriculture, computer science, science, health science and other disciplines. Now, you will discuss further applications involving statistics.

Example

The number of people who paid a tax were registered for 80 days and the data are as follows:

74	96	101	83	105	83	88	84	89	74
100	101	93	89	89	94	76	94	98	92
110	82	81	109	75	70	83	74	104	64
85	73	74	84	111	94	98	98	95	102
64	64	91	115	76	63	61	100	84	94
70	89	75	92	83	96	84	73	96	109
65	100	95	60	98	98	73	64	76	81
84	113	99	78	73	73	94	90	89	75

a. Construct a grouped frequency distribution with 10 classes.

b. Find the mean, mode and median from the grouped frequency distribution.

c. Find Q_2, D_5, D_7, P_{50}, and P_{75}.

Solution

a. **Table 7.19**

Class interval	No of days(f)	m_i	fm_i	Cumulative frequency (Cf)
60 – 65	8	62.5	500	8
66 – 71	2	68.5	137	10
72 – 77	15	74.5	1117.5	25
78 – 83	8	80.5	644	33
84 – 89	12	86.5	1038	45
90 – 95	12	92.5	1110	57
96 – 101	14	98.5	1379	71
102 –107	3	104.5	313.5	74
108 – 113	5	110.5	552.5	79
114 – 119	1	116.5	116.5	80

b.

$$\bar{x} = \frac{f_1m_1 + f_2m_2 + f_3m_3 + f_4m_4 + f_5m_5 + f_6m_6 + f_7m_7 + f_8m_8 + f_9m_9 + f_{10}m_{10}}{f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 + f_8 + f_9 + f_{10}} = \frac{6908}{80} = 86.35$$

This means, the mean number of people who paid a tax is 86.35.

The median class is that class containing the $\left(\frac{80}{2}\right)$ item = 40^{th} item.

It is in the 5^{th} class.

Therefore, the median class is **84 – 89.**

Thus, B_L = **84**, $\frac{n}{2} = 40$, $fc = 12$, $i = 6$, $cf_B = 33$, Therefore,

$$\tilde{X} = B_L + \left(\frac{\frac{n}{2} - cf_B}{fc}\right)i$$

$$= 84 + \left(\frac{40 - 33}{12}\right)6 = 84 + 3.5 = 87.5$$

Thus, the median of the number of the people who paid a tax is 87.5.

72 – 77 is the modal class, since it is a class with the highest frequency.

$B_L = 72$ $\quad d_1 = 15 - 2 = 13$ $\quad d_2 = 15 - 8 = 7$ and $i = 6$

$$\hat{X} = B_L + \left(\frac{d_1}{d_1 + d_2}\right)i$$

$$= 72 + \left(\frac{13}{20}\right)6 = 75.9$$

Hence, the mode of the number of people who paid a tax is 75.9

c. $Q_2 = D_5 = P_{50} = 87.5$

This means, in 50% of the days, the number of people who paid a Tax fall below 87.

The 5th decile and the 50th percentile is also the median of the number of people who paid a tax a day. Looking at the data in the Table 7.19, the median can be calculated as 87.5 = median = $D_5 = P_{50}$. At this point, in half of the days, the number of the people who paid a tax lies above and below 87.5.

D_7 is $\left(\frac{7 \times 80}{10}\right)^{th}$ item, that is, 56th item which lies in the 6th class. $cf_B = 45$, $f_C = 12$ and $i = 6$

$$D_7 = B_L + \left(\frac{\frac{7n}{10} - cf_B}{fC}\right)i = 90 + \left(\frac{\frac{7 \times 80}{10} - 45}{12}\right)6 = 95.5$$

This means, in 70% of the days, the number of the people who paid a tax lies below 95.5.

P_{75} is $\left(\frac{75 \times 80}{100}\right)^{th}$ item, that is, 60th item which lies in the 7th class. $cf_B = 57$, $f_C = 14$ and $i = 6$

$$P_{75} = B_L + \left(\frac{\frac{75n}{100} - cf_B}{fC}\right)i = 96 + \left(\frac{\frac{75 \times 80}{100} - 57}{14}\right)6 = 97.29$$

This means, in 75% of the days, the number of the people who paid a tax lies below 97.29.

Exercise 7.17

1. The lifespans, in days, of the bees in a colony are recorded.

Lifespan(days)	1–20	21 – 40	41 – 60	61 – 80	81 – 100
Number of bees	6	11	22	32	29

   a. Calculate the mean for lifespan of the bees.
   b. State the modal number of days.
   c. Find Q_2, D_6, P_{25}

2. The table below shows the frequency distribution of the mass of 200 rebars that have to be transported from the warehouse to the construction site. Let m be the mass of the rebar.

Mass(kg)	Number of rebars
$10 \leq m < 20$	32
$20 \leq m < 30$	38
$30 \leq m < 40$	64
$40 \leq m < 50$	35
$50 \leq m < 60$	22
$60 \leq m < 70$	9

Calculate the mean, median and mode mass of the steel bars.

Problem Solving

1. A house building company wanting to find out what type of houses they should build most often in a region carried out a survey in that region to find out the number of people in a family. Will they use mean, median or mode to decide what type of houses should be built most?
2. A car battery factory wants to give a warranty to their customers as to the lifetime of their batteries. i.e., they want to tell the customer if you have a problem with the battery in the next months, we will replace your battery with new one. They checked the lifetime of 100 batteries. Will they use mean, median or mode to decide on the number of months to guarantee their batteries?
3. In a mathematics test, Tolosa got 62%. For the whole class, the mean mark was 64% and the median mark was 59%. Which average tells Tolosa whether he is in the top half or the bottom half of the class?
4. The mean age of three people is 22 and their median age is 20. The range of their ages is 16. How old is each person?

Summary

1. Quantitative data can be quantified in terms of number value.
2. Qualitative data cannot be quantified as number value and expressed as quality.
3. The characteristics of a unit being observed that may assume more than one of a set of values to which a numerical measure can be assigned is variable.
4. Discrete data is a numerical type of data that includes whole, concrete numbers with specific and fixed data values determined by counting.
5. Continuous data is one which takes any fractional point along a specified interval of values.
6. Frequency means the number of times a certain value of a variable is repeated in the given data.
7. A grouped frequency distribution is constructed to summarize a large sample of data.
8. The appropriate class width is given by

$$\text{Class width} = \frac{\text{Largest Value in ungrouped data} - \text{Smallest Value in ungrouped data}}{\text{number of classes required}}.$$

9. A measure of location is a single value that is used to represent a mass of data. The common measures of location are mean, median, mode, quartiles, deciles and percentiles.

$$\text{Mean} = \overline{X} = \frac{x_1 + x_2 + x_3 + \dots + x_n}{n} \text{ for raw data.}$$

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + f_3x_3 + \dots + f_nx_n}{f_1 + f_2 + f_3 + \dots + f_n} \text{ for ungrouped data.}$$

$$\bar{x} = \frac{f_1m_1 + f_2m_2 + f_3m_3 + \dots + f_im_i}{f_1 + f_2 + f_3 + \dots + f_i} \text{ for grouped data.}$$

10. Mode is the value with the highest frequency.
11. If a distribution has a single mode, it is "unimodal".
    If it has two modes, it is "bimodal".
    If it has more than two modes, it is called "multimodal".

12. Median of ungrouped data is given by

$\tilde{X} = \left(\frac{n+1}{2}\right)^{th}$ item, if n is odd. $\tilde{X} = \dfrac{\left(\frac{n}{2}\right)^{th}\text{item}+\left(\frac{n}{2}+1\right)^{th}\text{item}}{2}$, if n is even.

13. Median of grouped data is given by $\tilde{X} = B_L + \left(\dfrac{\frac{n}{2} - cf_B}{fc}\right)i$

14. Quartiles for ungrouped frequency distributions are given by

i) $Q_k = \left(\frac{k(n+1)}{4}\right)^{th}$ item; if number of observations is odd.

ii) $Q_k = \left(\dfrac{\left(\frac{kn}{4}\right)+\left(\frac{kn}{4}+1\right)}{2}\right)^{th}$ item; if number of observations is even.

15. Decile for ungrouped frequency distributions are given by

i) $D_j = \left(\frac{j(n+1)}{10}\right)^{th}$ item; if number of observations is odd.

ii) $D_j = \left(\dfrac{\left(\frac{jn}{10}\right)+\left(\frac{jn}{10}+1\right)}{2}\right)^{th}$ item; if number of observations is even.

16. Percentile for ungrouped frequency distributions are given by

i) $P_j = \left(\frac{j(n+1)}{100}\right)^{th}$ item; if number of observations is odd.

ii) $P_j = \left(\dfrac{\left(\frac{jn}{100}\right)+\left(\frac{jn}{100}+1\right)}{2}\right)^{th}$ item; if number of observations is even.

17. Quartiles for grouped frequency distributions are given by

$$Q_K = B_L + \left(\frac{\frac{kn}{4} - cf_B}{fk}\right)i.$$

18. Decile for grouped frequency distributions are given by

$$D_j = B_L + \left(\frac{\frac{jn}{10} - cf_B}{fc}\right)i.$$

19. Percentile for grouped frequency distributions are given by

$$P_j = B_L + \left(\frac{\frac{jn}{100} - cf_B}{fc}\right)i.$$

20. The mode is given by

Mode $= \hat{X} = B_L + \left(\dfrac{d_1}{d_1 + d_2}\right)$ i, for grouped frequency distributions.

Review Exercise

1. Construct a grouped frequency distribution table and histogram for the following data using 6 classes:

61	83	92	87	74	60	71	57	61	69
85	40	85	59	80	52	56	81	77	79
78	82	43	64	67	48	81	68	37	43
65	85	49	69	61	54	56	69	68	78
74	59	76	65	69	49	57	38	73	81

2. Find mean, mode(s) and median of each of the following scores:

   a. 12, 13, 26, 15, 25, 38, 28, 27, 32

   b. 8, 7, 11, 13, 17, 9, 8, 7, 6, 11

   c.

Scores(x)	22	23	24	26	29	36
f	6	13	30	23	17	9

3. The following table gives the daily income of 100 workers of a factory: Let x be the daily income.

Daily income(birr)	Number of Workers
$200 \leq x < 220$	18
$220 \leq x < 240$	28
$240 \leq x < 260$	21
$260 \leq x < 280$	21
$280 \leq x < 300$	12

Find the mean, median and mode of the above data.

4. In a small business 2 cleaners earn Birr 340 each, 6 persons handling the machinery earn Birr 600 each, the manager earns Birr 1500 and the director Birr 3500 per month. Which measure, mean, median or mode best represents these data?

5. The number of people who paid a tax were registered for 80 days and the data are as follows:

74	96	101	83	105	83	88	84	89	74
100	101	93	89	89	94	76	94	98	92
110	82	81	109	75	70	83	74	104	64
85	73	74	84	111	94	98	98	95	102
64	64	91	115	76	63	61	100	84	94
70	89	75	92	83	96	84	73	96	109
65	100	95	60	98	98	73	64	76	81
84	113	99	78	73	73	94	90	89	75

a. Construct a grouped frequency distribution, with 10 classes.

b. Find the mean, mode and median from the data.

c. Find Q_2, D_5, D_7, P_{50}, and P_{75}.

UNIT 8

PROBABILITY

Unit Outcomes

By the end of this unit, you will be able to:

- Distinguish certain and uncertain events.
- Know principles of counting.
- Explain the concept of probability.
- Understand Binomial Theorem.
- Calculate the probability of an event.
- Calculate the probability of a compound event.
- Apply facts and principles in computation of probability.
- Represent probabilities as fractions, decimals and percentages.
- Interpret probabilities as fractions, decimals and percentages.
- Represent the probability of an event as fraction or decimal between 0 and 1 or as a percentage.
- Use probability concept to solve real life problems.

Unit Contents

KEY Words

- **combination**
- **dependent events**
- **exhaustive events**
- **fundamental counting principles**
- **independent events**
- **permutation**
- **probability of an event**

8.1 Introduction

From the time you awake until you go to bed, you make decisions regarding the possible events that are governed at least in part by chance. For example, should you carry an umbrella to school today? Should you accept that new home work? Thus, probability can be defined as the chance of an event occurring. Many people are familiar with probability from observing or playing games of chance, such as card games, slot machines, or lotteries. In addition to being used in games of chance, it is used in the fields of insurance, investments, and weather forecasting and in various other areas. Finally, probability is the basis of inferential statistics. For example, predictions are based on probability and hypotheses are tested by using probability.

Blaise Pascal (1623 – 1662) suffered the most appalling ill-health throughout his short life. He is best known with Fermat on probability. This followed correspondence with a gentleman gambler who was puzzled as to why he lost so much in betting on the basis of appearance of a throw of dice. Pascal's work on probability become of enormous importance and showed for the first time that absolute certainty is not a necessity in mathematics and science. He also studied physics, but his last years were spent in religious meditation and illness.

Source: Cambridge IGCSE Mathematics Fifth Edition.

Revision about Probability

You will revise some terms before you proceed to the next section as discussed in grade 9.

1. An **experiment** is an activity (measurement or observation) that generates well-defined results (Outcomes).
2. An **outcome (Sample point)** is the result of a single trial of an experiment.
3. A **sample space (S)** is a set that contains all possible outcomes of an experiment.
4. An **Event** is any subset of a sample space.

Example 1

If a fair die is rolled once the possible results are either 1, 2, 3, 4, 5 or 6, then from this:

a. Give the sample space.
b. Give the event of obtaining odd numbers.
c. Give the event of obtaining even numbers.
d. Give the event of obtaining number seven.

Solution

a. sample space(S) is {1, 2, 3, 4, 5, 6}.

b. the event of obtaining odd numbers is {1, 3, 5}.

c. the event of obtaining even numbers is {2, 4, 6}.

d. the event of obtaining number seven is $\emptyset$; which is an impossible event.

Example 2

When a "fair" coin is tossed, the possible results are either head (H) or tail (T). Consider an experiment of tossing a fair coin three times.

a. What are the possible outcomes?

b. Give the sample space.

c. Give the event of H appearing on the second throw.

d. Give the event of at least one T appearing.

Solution

a. HHH, HHT, HTH, THH, TTH, THT, HTT, TTT.

b. S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}.

c. A = {HHH, HHT, THH, THT}.

d. B = {HHT, HTH, THH, TTH, THT, HTT, TTT}.

Note

Events which have the same chance of occurring are **equally likely events.**

Exercise 8.1

1. A fair die is rolled once:

   a. Give the event of obtaining a number equal to or greater than 4.

   b. Give the event of obtaining number zero.

2. A fair coin is tossed twice:
   a. What are possible outcomes?
   b. Give the sample space.
   c. Give the event of at least one H appearing.

8.2 Fundamental Principle of Counting

Activity 8.1

1. Consider that six students leave the class; re-enter one by one and take a seat. In how many ways can you seat these students?
2. Consider students observing car plate numbers. The first digit is from 26 English alphabet, the other five digits are a number from 0 to 9. Discuss in how many different ways a car plate number is written.

From Activity 8.1, in order to calculate probabilities, you have to know the number of elements of an event and the number of elements of the sample space. Thus, if the experiment is leaving and re-entering of 6 students into the class, what is the total number of possible outcomes? If an event E is defined by number of ways the students can seat, similarly if the experiment is car plate number, what is the total number of possible outcomes? If an event E is defined by "6 English alphabet and 6-digit number", then how do you find n(E)? From this, you can observe that counting plays a very important role in finding probabilities of events.

Thus, when the number of possible outcomes is very large, it will be difficult to find the number of possible outcomes by listing. One may use fundamental principle of counting which helps us to find the number of ways of occurrence (selections) of events in a given order such as the multiplication and the addition principle.

8.2.1 Multiplication Principle

Consider the following example before stating the multiplication principle.

Example 1

Consider that Girma goes to the nearest snack to have breakfast. He can take tea, coffee, or milk with bread, cake, or sandwich. How many possibilities does he have?

Solution

As shown below, use a tree diagram to show the number of events.

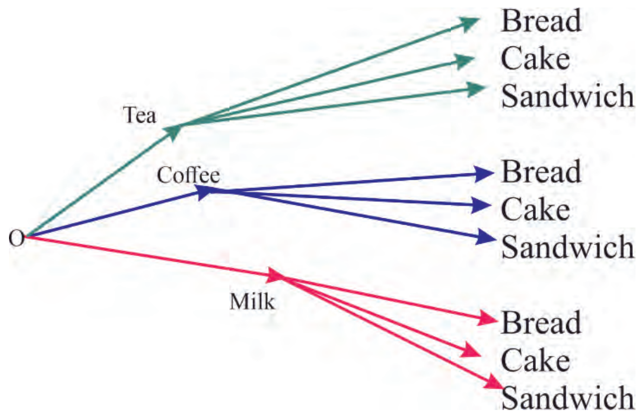


Figure 8.1

There are $3 \times 3 = 9$ possibilities as shown in Figure 8.1

From these possible choices, Girma can take tea with bread, tea with cake, tea with sandwich or coffee with bread, coffee with cake, coffee with sandwich or milk with bread, milk with cake, milk with sandwich.

The above example, therefore, illustrates the multiplication principle of counting.

Multiplication principle

If an event can occur in **m** different ways, and for every such choice another event can occur in **n** different ways, then both the events can occur in the given order in **m** × **n** different ways. That is, the number of ways in which a series of successive things can occur is found by multiplying the number of ways each thing can occur.

Example 2

Suppose there are 6 seats arranged in a row. In how many different ways can six people be seated on them?

Solution

The first man has 6 choices, the 2nd man has 5 choices, the 3rd man has 4 choices, the 4th has 3 choices, the 5th man has 2 choices and the sixth man only one choice.

Therefore, the total number of possible seating arrangements is

$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$

Example 3

A coin is tossed and a die is rolled. Find the number of outcomes for the sequence of events.

Solution

For a coin, 2 possible outcomes: H and T.

For a die, 6 possible outcomes :1, 2, 3, 4, 5 and 6.

Now when coin is tossed and a die is rolled simultaneously, $(2 \times 6) = 12$ possible outcomes.

Exercise 8.2

1. Abebech has got only clean clothes of 3 t-shirts and 5 pairs of jeans. How many different combinations can Abebech choose?
2. Suppose that a man has 5 coats, 10 shirts and 8 different trousers. In how many different ways can a man dress?
3. A paint manufacturer wishes to manufacture several different paints. The categories include:

Table 8.1

Category	Painting materials
Colour	Red, blue, white, black, green, brown, yellow
Type	Latex, oil
Texture	Flat, semigloss, high gloss
Use	Outdoor, indoor

How many different kinds of paint can be made if you can select one colour, one type, one texture, and one use?

4. There are 16 microcomputers in a computer center. Each microcomputer has 21 ports. How many different ports to a microcomputer in the center are there?

8.2.2 Addition Principle

Before stating the addition principle, let us see the following examples:

Example 1

Suppose that you want to buy a computer from one of two makes A_1 and A_2. Suppose also that those makes have 12 and 18 different models, respectively. How many models are there altogether to choose from?

Solution

Since you can choose one of 12 models of make A_1 or one of 18 of A_2, there are altogether $12 + 18 = 30$ models to choose from a given model.

Choosing one from given models of the either make is an **event** and the choices for either event is the **outcomes** of the event. Thus, the event "selecting one from make A_1", for example, has 12 outcomes. This shows the Addition Principle of Counting.

Addition principle

If A and B are mutually exclusive events (i.e., the occurrence of one excludes that of the other) where an event A can occur in **m** ways and another event B can happen in **n** ways, respectively, then the total number of outcomes for the event A or B is **n + m** ways.

Note

Two events are said to be mutually exclusive, if both cannot occur simultaneously.

Example 2

If 4 red and 6 green marbles are placed in a bag. How many marbles are there to choose from?

Solution

Since the red and green marbles cannot be chosen together or at the same time, the number of outcomes is added together. So, there are $4 + 6 = 10$ marbles to choose from.

Example 3

A question paper has two parts where one part contains 10 questions and the other 6 questions. If a student has to choose only one question from either part, in how many ways can the student do it?

Solution

The student can choose one question in $10 + 6 = 16$ ways.

Example 4

Suppose that either a member of the Mathematics faculty or a student who is Mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the Mathematics faculty and 83 Mathematics majors and no one is both a faculty member and a student?

Solution

There are 37 ways to choose a member of the Mathematics faculty and there are 83 ways to choose a student who is Mathematics major. Choosing a member of the Mathematics faculty is never the same as choosing a student who is Mathematics major because no one is both a faculty member and a student. By addition principle it follows that there are $37 + 83 = 120$ possible ways to pick this representative.

Example 5

Tigist is selecting an outfit. She has 5 different pairs of pants, 4 different skirts and 12 different T-shirts. How many outfits are possible?

Solution

Tigist cannot wear a pair of pants and a skirt at the same time. So, she can wear either "pants and T-shirt" or "skirt and T-shirt."

When she wears a pair of pants and a T-shirt, possible outfits are

$$5 \times 12 = 60$$

When she wears a skirt and a T-shirt, possible outfits are

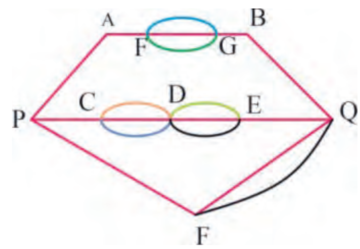
$$4 \times 12 = 48$$

Therefore, total possible outfits are

$$5 \times 12 + 4 \times 12 = (5 + 4) \times 12 = 108.$$

Exercise 8.3

1. Suppose there are 5 chicken dishes and 8 beef dishes. How many selections does a customer have?
2. There are 5 vegetarian entrée options and 7 meat entrée options on a dinner menu. What is the total number of entrée options?
3. Consider the following road system from P to Q. In how many different ways could one travel from P to Q?



4. Aster will draw one card from a standard deck of playing cards. How many ways can she choose:

   a. an even number? b. a king or a queen?

   c. a heart, a diamond or a club? d. a king or a black?

8.3 Permutations and Combinations

Factorial

Activity 8.2

1. What knowledge do you have about factorial of a number?
2. Compute a. 5! b. 9!

Definition 8.1

Factorial of a number denoted by n! is the product of all positive integers less than or equal to n:

$n! = n \times (n - 1) \times (n - 2) \times (n-3) \ldots 3 \times 2 \times 1$ and

$0! = 1 \quad 1! = 1$

Example 1

Compute

a. 4!   b. 7!   c. $\frac{10!}{8!}$ d. $\frac{45!}{42!}$ e. $\frac{8!}{5!\,2!}$

Solution

According to the definition 7.1 above:

a. $4! = 4 \times 3 \times 2 \times 1 = 24$.

b. $7! = 7\times6\times5\times4\times3\times2\times1 = 5040$.

c. $\frac{10!}{8!} = \frac{10 \times 9 \times 8!}{8!} = 90$.

d. $\frac{45!}{42!} = \frac{45 \times 44 \times 43 \times 42!}{42!} = 85,140$.

e. $\frac{8!}{5!\,2!} = \frac{8 \times 7 \times 6 \times 5!}{5!\,2!} = 168$.

8.3.1 Permutation

Permutations (1)

Activity 8.3

1. Compute the following:

   a. $\frac{7!}{(7-5)!}$ b. $\frac{11!}{(11-3)!}$ c. $\frac{9!}{(9-2)!}$

2. Consider a business owner who has a choice of 5 locations in which to establish her business. She decides to rank each location according to certain criteria, such as price of the store and parking facilities. Discuss the number of different ways she can rank the 5 locations.

From Activity 8.3, you can define permutation formally as follows.

Definition 8.2

The arrangement of n objects in a specific order using r objects at a time is called a permutation of n objects taking r objects at a time. It is written as P (n, r) or nPr, and the formula is

$$nPr = \frac{n!}{(n-r)!}, \text{ where } 0 < r \leq n.$$

This formula can be derived this way. Suppose there are five balls of different colours. You want to pick up three of them and line them up. How many different ways will you have to arrange the three balls? This answer is denoted as ${}_5P_3$. Let's calculate this. You pick up the first ball. You will have 5 choices. Then for second ball, you will have 4 balls to choose from. Similarly, for the third ball, you will have 3 remaining balls to choose from. Then the number of all possible ways is calculated as

$${}_5P_3 = 5 \times 4 \times 3 = 60$$

This is in fact the same as

$${}_5P_3 = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \frac{5!}{2!} = \frac{5!}{(5-3)!}$$

This was the case where n = 5 and r = 3. From this you can generalize that

$$nPr = \frac{n!}{(n-r)!}.$$

Note

The number of permutations of a set of n objects taken all together is denoted by P (n, n) or nPn and is equal to n!.

Thus, P (n , n) = n!

Example 2

Compute the following permutation:

a. P (7, 4) b. P (9, 6) c. P (14, 4)

Solution

a. $P(7, 4) = \frac{7!}{(7-4)!} = 840.$

b. $P(9, 6) = \frac{9!}{(9-6)!} = 60{,}480.$

c. $P(14, 4) = \frac{14!}{(14-4)!} = 24{,}024.$

Exercise 8.4

1. Find the factorial of each of the following numbers:

   a. 5! b. 8! c. 13!

2. Compute the following permutation:

   a. P (5, 1) b. P (5, 2) c. P (5, 3) d. P (5, 5)

Permutation (2)

Example 3

Five students are contesting an election for 5 places in the executive committee of environmental protection club in their school. In how many ways can their names be listed on the ballot paper?

Solution

You have to arrange 5 names in 5 places. Therefore, the number of ways of listing their names on the ballot paper, $P(5, 5) = 5! = 120$.

Example 4

Suppose you have letters A, B, C, and D:

a) How many permutations are there taking all the four?

b) How many permutations are there taking two letters at a time?

Solution

a. Here, n = 4, i.e., there are four distinct objects.

$\Rightarrow$ There are 4! = 24 permutations.

b. Here, n = 4, r = 2

$\Rightarrow$There are ${}_4P_2 = \frac{4!}{(4-2)!} = 12$ permutations.

Exercise 8.5

1. A mathematics debating team consists of 4 speakers:
   a. In how many ways can all 4 speakers be arranged in a row for a photo?
   b. How many ways can the leader and vice - leader be chosen?
2. 3 different statistics books, 5 different mathematics books, and 3 different physics books are arranged on a shelf. How many different arrangements are possible if;
   a. The books in each particular subject must all stand together
   b. Only the statistics books must stand together
3. R, S, T, U, V and W are six students. In how many ways can they be seated in a row if:
   a. there are no restrictions on the seating.
   b. R and S must sit beside each other.
   c. U, V and W must sit beside each other.
   d. R and W must sit at the end of each row.

Permutations with Repetition

When you arrange four letters, A, B, C, and D, you will have ${}_4P_4 = 4 \times 3 \times 2 \times 1 = 24$ different ways of arrangements. However, if you have four letters of A, B, C and C, how many different ways of arrangements will you have?

Suppose you can differentiate two C's as C_1 and C_2. Then, the total number of ways of arrangements is 24. In this calculation, ABC_1C_2 and ABC_2C_1 are different arrangements, but in fact you cannot differentiate those two permutations: ABCC and ABCC are the same arrangement. Similarly, BC_1AC_2 and BC_2AC_1 are the same arrangement, BCAC. Thus, when there are two C's, the total number of permutations, 24, indicates duplications. For instance, you have counted ABCC twice as ABC_1C_2 and ABC_2C_1. Since C_1 and C_2 can be arranged in 2! = 2 ways (C_1C_2 and C_2C_1), the total number of 24 should be divided by 2 to obtain the correct total number of arrangements, which is 12.

When there are some same objects included in the objects, the total number of arrangements of the objects should be counted avoiding duplications.

Definition 8.3

The number of arrangements of n objects in which n_1 are alike, n_2 are alike, …, n_r alike objects of the r^{th} type is $\frac{n!}{n_1! \times n_2! \times \ldots \times n_r!}$, where $n_1 + n_2 + \ldots + n_r = n$ and $0 < r \le n$.

Example 5

How many different permutations can be made from the letters in the word "ADDIS" ?

Solution

Here n = 5 of which there are two D's.

So, 1 is A, 2 are D, 1 is I and 1 is S.

$\Rightarrow n_1 = 1,\ n_2 = 2,\ n_3 = 1,\ n_4 = 1$

Using definition 8.3, there are

$$\frac{5!}{n_1! \times n_2! \times n_3 \times n_4!} = \frac{5!}{1! \times 2! \times 1! \times 1!} = \frac{5\times 4\times 3\times 2\times 1}{2\times 1} = 60 \text{ permutations.}$$

Example 6

How many different permutations can be made from the letters in the word "ADDIS" if the rearrange letters which start from S?

Solution

If you fix the first letter to be S, then you have four remaining letters, A, D, D, and I, to arrange. You have to think of the permutations of them. Since there are 2 D's, using definition 8.3, you have

$$\frac{4!}{1! \times 2! \times 1!} = \frac{4\times 3\times 2\times 1}{2\times 1} = 12 \text{ permutations:}$$

They are:

SADDI	SDADI	SIADD
SADID	SDAID	SIDAD
SAIDD	SDDAI	SIDDA
	SDDIA	
	SDIAD	
	SDIDA	

Exercise 8.6

1. Determine how many different permutations are possible using the word ABABA.
2. Find how many ways you can rearrange the letters of BANANA; if
   a. there is no restriction.
   b. the rearranged letters start with N.
3. How many different permutations can be made from the letters in the word "MATHEMATICS"?
4. How many different permutations can be formed from the letters of the word MATHEMATICS which start from C?

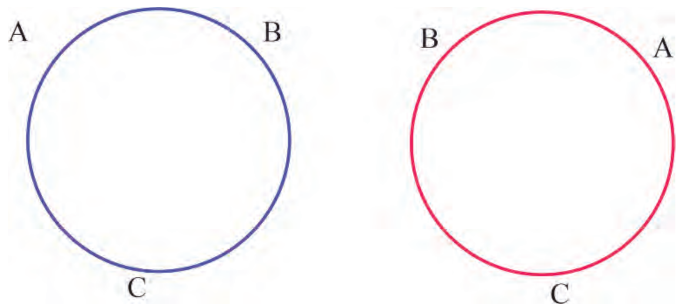
5. How many permutations can be made from the letters in the word MATHEMATICS which start with M?

Circular Permutations

Activity 8.4

Discuss the difference between arrangements of objects in a straight line and around a circle.

Suppose you have three students named A, B, and C. You have already determined that they can be seated in a straight line in 3! or 6 ways. your next problem is to see in how many ways these students can be seated in a circle. You draw a diagram.



It happens that there are only two ways we can seat three students in a circle, relative to each other's positions. This kind of permutation is called a circular permutation which depends on the relative positions of the students after you fix the position of one student. In such cases, no matter where the first student sits, the permutation is not affected. Each student can shift as many places as they like, and the permutation will not be changed. You are interested in the position of each student in relation to the others. Thus, in circular permutations, the first student is considered a place holder, and where he/she sits does not matter.

Definition 8.4

The number of permutations of n elements in a circle is $\frac{n!}{n} = (n-1)!$

Example 7

In how many different ways can five people be seated at a round table?

Solution

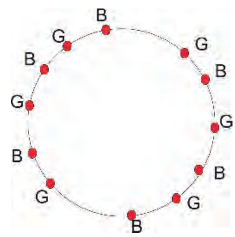
The number of ways will be $(5-1)!$ or 24.

Example 8

In how many ways 6 boys and 5 girls dine at a round circular table, if no two girls are to sit together.

Solution

First let us allot the seats to boys. Now 6 boys can have $(6-1)!$ circular permutation, i.e., the number of permutations in which boys can take their seats is $5! = 120$. Next the 5 girls occupy seats marked(G). There are 6 spaces between the boys, which can be occupied by 5 girls in ${}_6P_5 = 720$ ways. Hence total number of ways is



$5! \times {}_6P_5 = 120 \times 720 = 86{,}400$.

Example 9

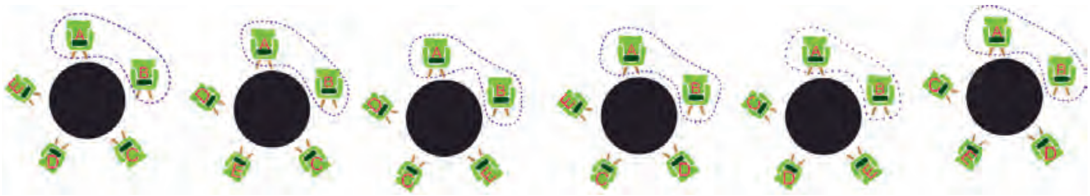
Find the number of ways in which 5 people A, B, C, D, and E can be seated at a round table, such that

a. A and B always sit together.

b. C and D never sit together.

Solution

a. If you wish to make seat A and B together in all arrangements, you can consider these two as one unit, along with 3 others. So, effectively you have to arrange 4 people in a circle, the number of ways being (4 – 1)! or 6. Let's take a look at these arrangements:



But in each of these arrangements, A and B can themselves interchange places in 2 ways as shown in the figures below.

Therefore, the total number of ways will be 6 × 2 or 12.

b. The number of ways in this case would be obtained by removing all those cases (from the total possible) in which C and D are together. The total number of ways will be (5 – 1)! or 24. Similar to (**a**) above, the number of cases in which C and D are seated together, will be 12. Therefore, the required number of ways will be 24 – 12 or 12.

Exercise 8.7

1. Calculate circular permutation of 4 persons sitting around a round table.
2. In how many ways can four couples be seated at a round table if the men and women want to sit alternately?
3. In how many ways 8 boys and 3 girls can sit around a circular table, so that no two girls sit together?

8.3.2 Combination

Combinations (1)

Activity 8.5

1. Compute the following:

   a. $\frac{P(7,3)}{3!}$ b. $\frac{P(14,7)}{7!}$ c. $\frac{P(n,r)}{r!}$ d. $\frac{P(n,n)}{n!}$

2. Suppose a dress designer wishes to select two colours of material to design a new dress, and he has on hand four colours. Discuss how many different possibilities can there be in this situation?

From Activity 8.5, you have observed that combination can be obtained by dividing permutation by factorial of r objects selected from n objects.

Example 1

Given the letters A, B, C, and D, list the permutations and combinations for selecting two letters.

Solution

The permutations are:

AB	BA	CA	DA
AC	BC	CB	DB
AD	BD	CD	DC

In permutations, AB is different from BA. But in combinations, AB is the same as BA since the order of the objects does not matter in combinations. Therefore, if duplicates are removed from a list of permutations, what is left is a list of combinations, as shown:

AB	~~BA~~	~~CA~~	~~DA~~
AC	BC	~~CB~~	~~DB~~
AD	BD	CD	~~DC~~

Hence, the combinations of A, B, C, and D are AB, AC, AD, BC, BD, and CD. (Alternatively, BA could be listed and AB crossed out, etc.). This example leads us to the definition of combinations.

Definition 8.5

The number of combinations of r objects selected from n objects without considering the order of selection, is denoted by $C(n, r) = {}_nC_r = \binom{n}{r} = C_r^n$ and defined by

$$C(n, r) = \frac{n!}{(n-r)!\,r!} \quad \text{where } 0 < r \leq n.$$

To arrive at a formula for nCr, observe that the *r* objects in nPr can be arranged among themselves in *r*! ways.

Thus, $C(n, r) = \frac{nPr}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{(n-r)!\,r!}$

Hence, the number of possible combinations of *n* objects taken *r* at a time is given by the formula

$$C(n, r) = \frac{n!}{(n-r)!\,r!} \quad \text{where } 0 < r \leq n.$$

From this, you can see that the number of ways that a committee of three members can be selected from four individuals is given by

$C(4, 3) = \frac{4!}{1!\,3!} = 4$ ways.

Example 2

Compute the following:

a. C (7, 3) b. C (11, 5) c. C (13, 10)

Solution

a. $C(7, 3) = \frac{7!}{(7-3)!\,3!} = \frac{7!}{4!3!} = \frac{7 \times 6 \times 5 \times 4!}{4! \times 3 \times 2 \times 1} = 35.$

b. $C(11, 5) = \frac{11!}{(11-5)! \times 5!} = \frac{11!}{6! \times 5!} = \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6!}{6! \times 5 \times 4 \times 3 \times 2 \times 1} = 462.$

c. $C(13, 10) = \frac{13!}{(13-10)!\,10!} = \frac{13 \times 12 \times 11 \times 10!}{3 \times 2 \times 1 \times 10!} = 286.$

Exercise 8.8

1. Compute each of the following.

   a. C (5, 2) b. C (5, 3) c. C (9, 1) d. C (11, 9) e. C (n, n)

2. C (5, 2) = C (5, 3). Explain in words why this happens.
3. If C (n, 3) = C (n, 4), find n.

Combinations (2)

Example 3

In an examination paper, there are 10 questions. In how many different ways can a student choose six questions in all?

Solution

The student is to choose 6 questions from the 10 questions. Their order does not matter.

Hence, he/she can do it in C (10, 6) ways. This means, $C(10, 6) = \frac{10!}{(10-6)!\,6!} = 210.$

Example 4

In HIV/AIDS club there are 7 women and 5 men. A committee of 3 women and 2 men is to be chosen. How many different possibilities are there?

Solution

Here, you must select 3 women from 7 women, which can be done in ${}_7C_3$, or 35, ways. Next, 2 men must be selected from 5 men, which can be done in ${}_5C_2$, or 10 ways.

Finally, by the fundamental counting rule, the total number of different ways is $35 \times 10 = 350$, since you are choosing both men and women separately. Using the formula gives

$${}_7C_3 \times {}_5C_2 = \frac{7!}{(7-3)!\,3!} \times \frac{5!}{(5-2)!\,2!} = 350.$$

Example 5

In how many ways can Bekele invite at least one of his friends out of 6 friends to an art exhibition?

Solution

At least one means that he can invite either one, two, three, four, five or all 6.

Therefore, the total number of ways in which he can invite at least one of his friends is given by (Addition principle)

$$C(6, 1) + C(6, 2) + C(6, 3) + C(6, 4) + C(6, 5) + C(6, 6)$$
$$= 6 + 15 + 20 + 15 + 6 + 1 = 63.$$

Exercise 8.9

1. Select 5 students from a class of 25 to write solutions to a homework problem on the board. If it doesn't matter who does which question, how many ways can these 5 students be picked?
2. A newspaper editor has received 8 books to review. He decides that he can use 3 reviews in his newspaper. How many different ways can these 3 reviews be selected?

3. A committee of 5 students has to be formed from 8 boys and 4 girls. In how many ways can this be done when the committee consists of:

   a. exactly 3 girls? b. at least 3 boys? c. at most 3 girls?

4. There are 7 women and 5 men in mathematics department:

   a. In how many ways can a committee of 4 people be selected?

   b. In how many ways can this committee be selected if there must be 2 men and 2 women on the committee?

   c. In how many ways can this committee be selected if there must be at least 2 women on the committee?

8.4 Binomial Theorem

Pascal's Triangle

Activity 8.6

1. Expand $(a + b)^0$, $(a + b)^1$ and $(a + b)^2$. Discuss the pattern in the expansions and its corresponding coefficients?
2. What is the coefficient of a^3b^2 after expanding $(2a + b)^5$?

From Activity 8.6, imagine the pattern in the expansions and its corresponding coefficients. Again, the expansion form of $(a + b)^n$ where n = 0, 1, 2 is

$(a + b)^0 = 1$

$(a + b)^1 = a + b$

$(a + b)^2 = a^2 + 2ab + b^2$

In these expansions, you observe that the total number of terms in the expansion is one more than the index. For example, in the expansion of $(a + b)^2$, number of terms is 3 whereas the index of $(a + b)^2$ is 2. Powers of the first quantity 'a' go on decreasing by 1 whereas the powers of the second quantity 'b' increases by 1, in the successive terms. In each term of the expansion, the sum of the indices of a and b is the same and is equal to the index of a + b.

Thus, arranging the coefficients in these expansions in the form of triangle defines Pascal's Triangle.

Definition 8.6

Pascal's triangle is a triangular arrangement of numbers that gives the coefficients in the expansion of any binomial expression, such as $(a + b)^n$ for $n = 0, 1, \ldots, n$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & \cdot & & & & & & \cdot & \\
 \cdot & & & & & & & & \cdot \\
\cdot & & & & & & & & \cdot
\end{array}
$$

Note

Pascal's triangle can be used to visualize many properties of the binomial coefficient and the binomial theorem.

Exercise 8.10

Using the Pascal's triangle, find the coefficients in the expansion of $(a + b)^n$

a. when $n = 5$. b. when $n = 6$.

Binomial theorem

For any positive integer n, the binomial expansion of $(a + b)^n$ is given by

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$$

Example 1

Expand $(a+b)^4$

$$(a+b)^4 = \binom{4}{0}a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^4$$

$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Example 2

Find the coefficient of a^3b^2 in the expansion of $(a+b)^5$

$$(a+b)^5 = \binom{5}{0}a^5 + \binom{5}{1}a^4b + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}ab^4 + \binom{5}{5}b^5$$

Thus, the coefficient of $a^3b^2 = \binom{5}{2} = \frac{5!}{2!\,3!} = 10.$

Example 3

Expand $(a-2b)^4$ using Pascal's triangle.

Solution

Pascal's triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Hence, $(a-2b)^4 = a^4 + 4(a^3)(-2b) + 6a^2(-2b)^2 + 4(a)(-2b)^3 + (-2b)^4$

$$= a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4$$

Exercise 8.11

1. Expand each of the following using the Binomial Theorem:

   a. $(a+b)^7$ b. $(a+b)^8$ c. $(a-3b)^9$

2. Without writing all the expanded terms, answer the following:

   a. What is the coefficient of a^5b^3 in the expansion of $(a+b)^8$?

   b. What is the coefficient of a^2b^4 in the expansion of $(a+b)^6$?

   c. What is the coefficient of the term containing a^4b^2 in $(2a+b)^6$?

3. Expand $(x-3)^4$ using pascal's triangle.

8.5 Random Experiments and Their Outcomes

Random Experiments (1)

Activity 8.7

Find the sample spaces for each of the following random experiments:

a. Tossing a coin

b. Tossing a pair of coins

c. Rolling a die

From Activity 8.7, you have observed that if the experiment is repeated under identical conditions, it does not necessarily produce the same results every time but the outcome in a trial is one of the several possible outcomes which can be defined as random experiment or probability experiment.

Definition 8.7

A **random experiment** is any well-defined procedure that produces an observable outcome that could not be perfectly predicted in advance.

Example 1

1. Tossing a fair coin is a random experiment.
2. Rolling of a die is a random experiment.
3. Drawing a card from a pack of cards is a random experiment.

Definition 8.8

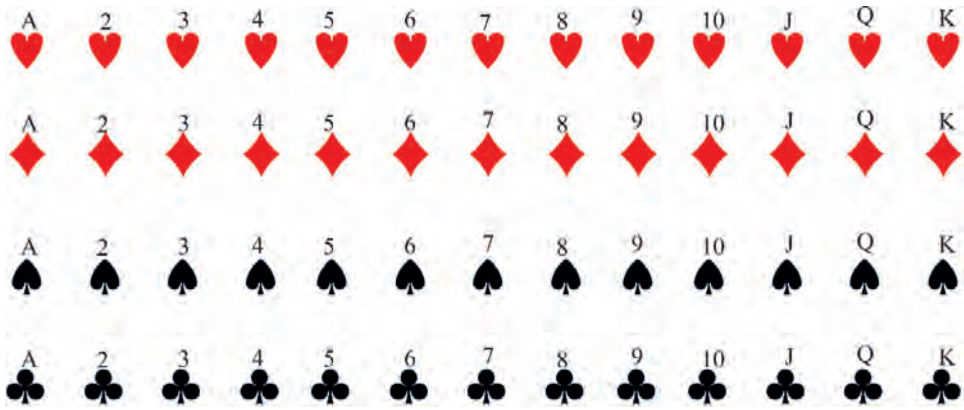
Sample space is the set of all possible outcomes of a random experiment associated with it and denoted by S.

Example 2

Find the sample spaces for drawing one card from an ordinary deck of cards.

Solution

There are 4 suits (hearts, clubs, diamonds, and spades) and 13 cards for each suit (ace through King). Thus, there are 52 outcomes in the sample space as shown below.



Example 3

Find the possible outcomes of selecting a 4-digit PIN from 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

Solution

There are 10 possible values for each digit of the PIN.

So, $10 \times 10 \times 10 \times 10 = 10^4 = 10{,}000$.

Example 4

A box contains 24 different balls. Find the number of possible outcomes when you select three balls from the box.

Solution

There are C (24, 3) = 2,024 number of possible outcomes.

Exercise 8.12

1. Find the sample space for the gender of the children if a family has four children.
2. A fair die is thrown. How many favourable outcomes are there for getting an even number.
3. Find the sample space for drawing one card from an ordinary deck of cards.
4. Find the possible outcomes of the football match between St. Gorge and Ethiopia Coffee (for each team, draw, loss or win).

Random Experiments (2)

Example 5

Each of 5 cards has one of the letters A, B, C, D and E on them. The cards are shuffled. Find number of different arrangements (possible outcomes).

Solution

Number of different arrangements = $5! = 5 \times 4 \times 3 \times 2 \times 1$.

Note

Outcomes of a random experiment are said to be **equally likely** when each element has **equal chance of being chosen**.

Example 6

In tossing coin, any one of the outcomes H or T, has an equal chance of appearing at the top. Thus, they are considered as **equally likely.**

Note

In a random experiment, the outcomes which insure the occurrence of a particular result is called **favourable outcomes** to that particular result.

Example 7

a. There are four blue marbles and one red marble in a jar. You pick up one marble from the jar. What is the number of favourable outcomes to get a red marble only.

b. If you draw one slip from a box that contains 12 slips of paper numbered 1 to 12, how many favourable outcomes are there for choosing a slip with an even number on it?

c. In picking a playing card from a pack of 52 cards, what is the number of favourable outcomes to get a picture card?

Solution

a. There is 1 favourable outcome. This is red marble.

b. There are 6 favourable outcomes. These are 2, 4, 6, 8,10 and 12.

c. There are 12 favourable outcomes. These are 4 Jacks, 4 Queens and 4 Kings.

Exercise 8.13

1. What is meant by equally likely?
2. Two balls are to be selected with replacement from a bag that contains one red, one blue, one green and one orange ball. Use the counting principle to determine the number of favourable outcomes to get blue.
3. Each of 5 cards has one of the letters A, B, C, D, and D on them. The cards are shuffled. What is the number of favourable outcomes that the letters A and B are together?
4. In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins 1,000,000 birrs. In this lottery, the order the numbers are drawn in doesn't matter. Find the number of favorable outcomes of the lottery drawn that you win one-million-birr prize if you purchase a single lottery ticket.

8.6 Events

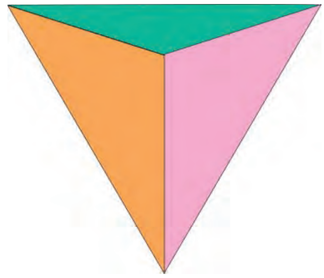
Events and Sample Space

Activity 8.8

1. What does an event mean?
2. List some events of the following experiments:
   a. Tossing a coin three times.
   b. Selecting a number at random from rolling a die.
   c. Drawing a ball from a bag containing 4 red and 6 white balls.

For Activity 8.8, remember that a subset of the sample space associated with a random experiment is called an event. It is denoted by E or other uppercase letters.

Example 1



The four faces of a regular tetrahedron are numbered 1, 2, 3 and 4. If it is thrown and the number on the bottom face (on which it stands) is registered, then list all the possible events of this experiment.

Solution

The sample space = {1, 2, 3, 4}.

Example 2

Suppose our experiment is tossing a fair coin. The sample space for this experiment is S = {H, T}. This sample space has a total of four possible events: a head is thrown ({H}), a tail is thrown ({T}), a head and a tail is thrown ({H}∩{T}= {} = Ø because this event is impossible) and a head **or** a tail is thrown ({H}∪{T} = {H, T} = S because this always happens).

Thus, the list of the possible events is Ø, {H}, {T}, and {H, T}.

Note

We can determine the possible number of events that can be associated with an experiment whose sample space is S. As events are subsets of a sample space, and any set with m elements has 2^m subsets, the number of events associated with a sample space with m elements is 2^m.

Example 3

Suppose our experiment is recording the gender of three children of three families. Where B and G standing for boy and girl. List events of the first result is a boy.

Solution

The sample space is S = {BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}.

E = {BBB, BBG, BGB, BGG} is an event.

Example 4

Consider the random experiment of throwing a die. List an event of getting a number(s) which is(are) less than 3.

Solution

The sample space = {1, 2, 3, 4, 5, 6}.

The possible events = {1, 2}.

Example 5

A committee of 5 people is to be selected from a group of 5 men and 6 women. What is the number of an event that the committee contains 2 men and 3 women?

Solution

Number of possible committee compositions = $C(11, 5) = 462$

Number of an event that the committee contains 2 men and 3 women is

$C(5, 2) \times C(6, 3) = 200$.

Exercise 8.14

1. Suppose you throw a fair die once. List an event of getting an odd number.
2. Suppose you throw a fair die twice and record the numbers. List an event of getting an odd number twice.
3. Three students are to be chosen from a class of 8 girls and 10 boys. What is the number of an event that the three selected students are two girls and one boy?

8.6.1 Types of Events

Various Events (1)

Activity 8.9

What do you know about types of events?

Definition 8.9

1. **Simple Event (Elementary Event)** is an event containing only one sample element.

Example 1

Suppose you randomly select one student from your class and observe whether the student selected each time is a man or a woman. The occurrence of man is a simple event.

2. **Compound Event**

An event that contains more than one sample element is called a compound event.

Example 2

When a die is rolled, if you are interested in the event of "getting an odd number", then the event will be a compound event, i.e., $\{1, 3, 5\}$.

3. **Impossible event**

An event that cannot happen is called an impossible event.

Example 3

If a die is thrown, then $S = \{1, 2, 3, 4, 5, 6\}$. Let E be the event of getting number 8, then E is an impossible event and denoted by $E = \emptyset$.

4. Certain or sure Event:

You know that every set is subset of itself. Thus, sample space (S) is a subset of itself and hence S is an event. This is called a Sure or certain event.

Example 4

If a die is thrown, then S = {1, 2, 3, 4, 5, 6}. Let E be the event of getting a number ≤ 6, then E = 1, 2, 3, 4, 5 or 6 is a sure or certain event.

5. Occurrence or Non-occurrence of an event

An event is said to occur if the outcome is associated to the event's sample space. Otherwise, it is a non-occurrence event.

Example 5

If a die is thrown, then S = {1, 2, 3, 4, 5, 6}. Let E be the event of getting an even number, then E = {2, 4, 6}. When you throw the die, if the outcome is 4, as $4 \in E$, then you say that E has occurred. If in another trial, the outcome is 3, then as $3 \notin E$, you say that E has not occurred (not E).

Exercise 8.15

You draw a card from a pack of 52 playing cards:

a. How many possible events are there?

b. Is the event that you draw 9 of spade a simple event or a compound event?

c. Is the event that you draw any card of spade a simple event or a compound event?

d. How do you call the event of drawing the card of 0 of heart?

Various Events (2)

6. Algebra of events

In a random experiment, let S be sample space, and let E_1 and E_2 be the events in S.

a. **Complement of an Event E_1**, denoted by E_1' (not E_1) consists of all events in the sample space that are not in E_1.

Example 6

In drawing a marble from a jar containing 6 red, 4 blue and 2 green marbles, let E be the event of green marble. Give the complement of the event.

Solution

Not E = E ' = 6 red and 4 blue marbles.

Example 7

Let a die be rolled once. Let E be the event of a prime number appearing at the top; i.e., E = {2, 3, 5}. Give the complement of the event.

Solution

E ' = {1, 4, 6}.

Note: $E' = S - E = \{a: a \in S \text{ and } a \notin E\}$.

a. Event E_1 or E_2 $= E_1 \cup E_2$ is an event that occurs when either one of E_1 or E_2 or both occur.

Example 8

In tossing 3 coins, let E_1 = having exactly one Head; i.e., {TTH, THT, HTT} and E_2 = having exactly one tail; i.e., {HHT, HTH, THH}, list events of $E_1 \cup E_2$.

Solution

$E_1 \cup E_2$ = {exactly one Head or exactly one tail}.

= {TTH, THT, HTT, HHT, HTH, THH}.

Event E_1 and $E_2 = E_1 \cap E_2$ is an event that occurs only when both E_1 and E_2 occurs.

Example 9

In throwing two dice,

Let E_1 = first die has 5 = {(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)}.

E_2 = sum on both dies is 6 = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)}.

List events of $E_1 \cap E_2$

Solution

Events of $E_1 \cap E_2$ = {(5, 1)}.

---

7. Exhaustive Events

A set of events is said to be exhaustive events if the performance of the experiment always results in the occurrence of at least one of them. Thus, if a set of events E_1, E_2, …, En are subsets of a sample space, S, they are said to be exhaustive,

if $E_1 \cup E_2 \cup \ldots, \cup E_n = S$.

Example 10

If a die is thrown, give instances of exhaustive events.

Solution

The sample space is S = {1, 2, 3, 4, 5, 6}. From this, the events E_1 = {2, 4},

E_2= {2, 4, 6} E_3= {1, 3, 5} are exhaustive events.

Since $E_1 \cup E_2 \cup E_3 = \{2, 4\} \cup \{2, 4, 6\} \cup \{1, 3, 5\} = \{1, 2, 3, 4, 5, 6\} = S$.

---

8. Mutually Exclusive Events

Two events E_1 and E_2 are said to be mutually exclusive events if the occurrence of any one of them excludes the occurrence of the other event. Thus, they cannot occur simultaneously.

Example 11

Say whether or not the following are mutually exclusive events.

a. When a coin is tossed once, the events {H} and {T}.

b. When a die is thrown, E_1 = getting an even number.

E_2 = getting 1 and 3.

c. When a die is rolled, E_1 = getting an odd number.

E_2 = getting a prime number.

d. When a card is drawn, the events of Kings and Aces.

Solution

a. Either you get head or tail but we cannot get both at the same time. Thus, {H} and {T} are mutually exclusive events. Since $E_1 \cap E_2 = \emptyset$.

b. E_1 and E_2 are mutually exclusive because $E_1 \cap E_2 = \{2, 4, 6\} \cap \{1, 3\} = \emptyset$.

c. E_1 and E_2 are not mutually exclusive because 5 is odd and prime at the same time.

d. Kings and Aces are mutually exclusive events.

9. Exhaustive and Mutually Exclusive Events

If S is a sample space associated with a random experiment and if $E_1, E_2, ..., E_n$ are subsets of S such that

a. $E_i \cap E_j = \emptyset$ for $i \neq j$ and,

b. $E_1 \cup E_2 \cup \ldots \cup E_n = S$ then the collection of the events $E_1, E_2, \ldots, E_n$ forms a mutually exclusive and exhaustive set of events.

Example 12

If a die is thrown, the events {2, 4, 6}, and {1, 3, 5} are mutually exclusive and exhaustive events. But, the events {1, 2}, {2, 3}, {4, 5, 6} are not because

$\{1, 2\} \cap \{2, 3\} = \{2\} \neq \emptyset$.

10. Independent Events

Events are said to be independent, if the occurrence or non-occurrence of one does not affect the occurrence or non-occurrence of the other.

Example 13

In a simultaneous throw of two coins, the event of getting a tail on the first coin and the event of getting a tail on the second coin are independent.

Example 14

a. Landing on heads after tossing a coin and rolling a 5 on a single 6-sided die.

b. Choosing a marble from a jar and landing on heads after tossing a coin.

c. Choosing a 3 from a deck of cards, replacing it, and then choosing an ace as the second card.

d. Rolling a 4 on a single 6-sided die, and then rolling a 1 on a second roll of the die.

11. Dependent Events

Events are said to be dependent, if the occurrence or non- occurrence of one event affects the occurrence or non-occurrence of the other.

Example 15

If a card is drawn from a well shuffled a pack of cards and the card is not replaced, then the result of drawing a second card is dependent on the first draw.

Exercise 8.16

When two dice thrown, consider following events:

E_1 = getting a prime number and E_2 = getting an even number.

a. List E_1 and E_2.

b. Are E_1 and E_2 exhaustive events?

c. Are E_1 and E_2 mutually exclusive events?

8.7 Probability of an Event

8.7.1 Revision on Probability

In this unit, you will revise the classical and the experimental approach of an event before you proceed to the next section as discussed in grade 9 such as:

1. Classical (mathematical) approach, and
2. Empirical (relative frequency) approach before you proceed to the next section.

1. Classical (Mathematical) Approach (1): Simple cases

Definition 8.10

In the classical approach to probability, the probability of an event occurring E is defined as the number of elements of the sample space included in the event, divided by the total number of elements in the sample space, when all outcomes of a random experiment are equally likely and mutually exclusive.

i.e., $P(E) = \frac{\text{Number of outcomes favourable to E}}{\text{Total number of outcomes}} = \frac{n(E)}{n(S)}$.

Example 1

A fair die is tossed once. Find the probability of getting:

a. Number 5.

b. a number greater than or equal to 4.

c. number 7.

Solution

First identify the sample space, say S

$S = \{1, 2, 3, 4, 5, 6\}$

$\Rightarrow n(S) = 6$

a. Let E be the event of number 5

$E = \{5\}$

$\Rightarrow n(E) = 1$

Thus, $P(E) = \frac{n(E)}{n(S)} = \frac{1}{6}$.

b. Let E be the events of a number greater than or equal to 4

$E = \{4, 5, 6\}$

$\Rightarrow n(E) = 3$

$P(E) = \frac{n(E)}{n(S)} = \frac{3}{6} = 0.5$.

c. Let E be the events of number 7

$E = \emptyset$

$\Rightarrow n(E) = 0$

$P(E) = \frac{n(E)}{n(S)} = \frac{0}{6} = 0$.

Exercise 8.17

1. If a die is rolled once, find the probability of getting an even number.
2. If a die is rolled twice, then find the following probabilities.
   a. The probability of getting the sum 2.
   b. The probability of getting the sum greater than 6.
   c. The probability of getting the sum greater than 9 or an odd.

Classical (mathematical) Approach (2): Cases using Permutations and Combinations

There are cases in which permutations and combinations are needed to calculate the number of outcomes.

Example 2

There are 4 blue balls and 3 red balls in a bag. You pick up 3 balls at random. Find the probability of getting 2 blue balls and 1 red ball.

Solution

First, calculate the total number of possible outcomes. You pick up 3 balls from 7 balls (their order does not matter), so the number of possible outcomes is given by

$${}_7C_3 = \frac{7!}{(7-3)! \times 3!} = \frac{7\times6\times5\times4\times3\times2\times1}{(4\times3\times2\times1)\times(3\times2\times1)} = 7 \times 5 = 35.$$

Second, calculate the number of possible outcomes where 2 blue balls and 1 red ball are chosen. Since you select 2 blue balls from 4 (their order does not matter), there are ${}_4C_2$ possible outcomes for blue balls. For red balls, there are ${}_3C_1$ possible outcomes. Since picking up blue balls and picking up red balls are independent, the number of possible outcomes of picking up 2 blue balls and 1 red ball is given by

$${}_4C_2 \times {}_3C_1 = \frac{4!}{2!2!} \times \frac{3!}{2!1!} = 6 \times 3 = 18.$$

Therefore, the probability is given by $\frac{{}_4C_2 \times {}_3C_1}{{}_7C_3} = \frac{18}{35}$.

Exercise 8.18

1. A box of 10 candles consists of 3 defective and 7 non-defective candles. If 5 of these candles are selected at random, what is the probability in which:
   a. 3 will be defective?
   b. 4 will be non-defective?
   c. all will be non-defective?

2. If 3 books are picked at random from a shelf containing 4 math books, 3 books of chemistry, and a dictionary, what is the probability that:
   a. the dictionary is selected?
   b. 2 math and 1 book of chemistry are selected?
3. Three cars are chosen at random from a certain car station containing 8 defective and 12 non-defective cars. What is the probability that?
   a. all are defective?
   b. all are non-defective?
   c. two are defective and the other is non- defective?

2. Empirical (relative frequency) approach

This approach is based on the observations obtained from random experiment. The empirical frequency of an event E is the relative frequency of event E.

Thus, $P(E) = \frac{\text{frequency of E}}{\text{total number of observations}} = \frac{f_E}{n}$.

Example 3

A travel agent determines that in every 70 reservations she makes 14 will be for a cruise. What is the probability that the next reservation she makes will be for a cruise?

Solution

Let E be the event of cruise

$P(E) = \frac{f_E}{n} = \frac{14}{70} = 0.20.$

Example 4

If records show that 30 out of 50,000 bulbs produced are defective, find the probability of a newly produced bulb to be defective.

Solution

Let E be the event that the newly produced bulb is defective.

$P(E) = \frac{f_E}{n} = \frac{30}{50,000} = 0.0006.$

Exercise 8.19

1. In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had type AB blood. Find the following probabilities:
   a. A person has type O blood.
   b. A person has type A or type B blood.
   c. A person has neither type A nor type O blood.
   d. A person does not have type AB blood.
2. Ten of the 500 randomly selected cars manufactured at a certain auto factory are found to be lemons (defective). Assuming that the lemons are manufactured randomly, what is the probability that the next car manufactured at this auto factory is a lemon?

8.7.2 The Axiomatic Approach of Probability

This approach includes both the classical and empirical definitions of probability.

Let **A** be a random experiment and S be a sample space associated with **A**. With each event **E** a real number called the probability of E, denoted by P(E), that satisfies the following properties called axioms of probability or postulates of probability:

1. $P(E) \geq 0$.
2. $P(S) = 1$, if $E = S$ (the sure or certain event).
3. $P(E_1 \cup E_2) = P(E_1) + P(E_2)$ If E_1 and E_2 are mutually exclusive events.
4. $0 \leq P(E) \leq 1$; i.e., the probability of an event is always between 0 and 1.
5. $P(\emptyset) = 0$ if $E = \emptyset$ (the impossible event)

6. If $E \cup E' = S$ then $P(E \cup E') = P(S) = 1$, and $P(E') = 1 - P(E)$, where $E' = S - E$. (not E); i.e., the sum of the probability of occurrence event (E) and non-occurrence event (E ') is 1, $(P(E) + P(E') = 1)$.

Note

Probability (P) is a function whose domain is the set of subsets of S (Sample space) and whose range is the set of real numbers between 0 and 1(both inclusive).

Example 1

A box contains 8 white balls. One ball is drawn at random. Find the probability of getting:

a. white ball   b. red ball

Solution

a. The box contains all white balls. Hence, we are sure that white will occur. Then, the probability of getting a white ball is one.

Thus, $P(w) = \frac{n(w)}{n(s)} = \frac{8}{8} = 1$.

b. The box contains no red balls. It is impossible to get a red ball and the probability is zero.

Hence, $P(R) = \frac{n(R)}{n(S)} = \frac{0}{8} = 0$.

Example 2

A bag contains 5 red, 7 black, and 6 white marbles. One marble is drawn at random. What is the probability that the marble is

a. black   b. not black

Solution

a. P(black) = $\frac{7}{18}$.

b. P (not black) = 1 − P(black)

$$= 1 - \frac{7}{18} = \frac{11}{18}$$

Thus, P(black) + P(not black) = $\frac{7}{18} + \frac{11}{18} = 1$.

Example 3

If one card is drawn from a deck, find the probability of picking these results:

a. King b. Not king c. King and Queen d. King or Queen

Solution

Number of ways king can happen, n(king) = 4 (there are 4 kings)

Number of ways Queen can happen, n(Queen) = 4 (there are 4 Queens)

Total number of outcomes, n(S) = 52 (there are 52 cards in total)

a. P(king) = $\frac{n(\text{king})}{n(S)} = \frac{4}{52} = \frac{1}{13}$.

b. P (Not king) = $1 - \frac{4}{52} = 1 - \frac{1}{13} = \frac{12}{13}$.

c. A card cannot be a king and a Queen at the same time. The chance of picking a King and a Queen at the same time is zero (or none).

Hence, P (a king and a Queen) = $\frac{0}{52} = 0$

d. P (a king or a Queen) = P (a king) + P (a Queen)

$$= \frac{1}{13} + \frac{1}{13} = \frac{2}{13}.$$

Example 4

Which of the following cannot be valid assignments of probabilities for outcomes of sample space $S = \{ a_1, a_2, a_3, a_4, a_5, a_6, a_7 \}$

	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a	0.2	0.001	0.09	0.03	0.01	0.008	0.3
b	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
c	0.1	0.2	0.3	0.4	0.5	0.6	0.7
d	– 0.7	0.007	0.3	0.4	– 0.2	0.1	0.3
e	$\frac{1}{13}$	$\frac{2}{13}$	$\frac{3}{13}$	$\frac{4}{13}$	$\frac{5}{13}$	$\frac{6}{13}$	$\frac{7}{13}$

Solution

a. Valid because all properties are satisfied.

b. not valid because the sum of the properties is $\frac{7}{6}$ which is greater than 1.

c. not valid, because the sum of all the probabilities is 2.8 which is greater than 1 i.e., $0 \leq P(E) \leq 1$ is not satisfied.

d. not valid, because probabilities of w_1 and w_5 are negative and hence $0 \leq P(E) \leq 1$ is violated.

e. not valid, because the sum of all the probabilities, $\frac{28}{13}$ is greater than 1.

Exercise 8.20

1. If one card is drawn from a deck, find the probability of getting these results:
   a. A queen
   b. A club
   c. not a club
   d. A 6 or a spade
   e. A red card and a 7
2. If two dice are rolled one time, find the probability of getting these results:
   a. A sum of 9
   b. A sum of 7 or 11
   c. A sum less than 9
   d. A sum greater than or equal to 10

8.7.3 Odds in Favour of and Odds Against an Event

Activity 8.10

1. Find the smallest share if you divide the number below by the ratio 2: 3

   a. 20 b. 35 c. 42

2. Find the largest share if you divide the number below by the ratio 2: 3

   a. 20 b. 35 c. 42

From Activity 8.10, you can formally define odds in favor of and odds against an event as follows:

Definition 8.11

If **P** and **P'** are probability of the occurrence and non- occurrence of an events respectively, then the ratio of **P: P '** is called **the odds in favour of the event** and the ratio of **P ': P** is called **the odds against the event**.

Note

If the probability of an event occurring is P, then the probability of the event not occurring (P ') is 1− P.

Example 1

If a race horse runs 100 races and wins 25 times and loses the other 75 times, what are the probability of winning and the odds of the horse winning?

Solution

The probability of winning is $\frac{25}{100} = 0.25$ and the probability of losing is $\frac{75}{100} = 0.75$.

The odds in favor of winning is 0.25: 0.75 $= \frac{25}{75} = 0.333$.

Example 2

The odds against certain events are 6: 8. Find the probability of its occurrence.

Solution

Let E be the event. Then we are given that n (E ') = 6 and n(E) = 8

Thus, n(S) = n (E') + n(E) = 6 + 8 = 14

Thus, $P(E) = \frac{n(E)}{n(S)} = \frac{8}{14} = \frac{4}{7}$.

Example 3

In throwing a die,

a. Find the odds in favor of getting 3 dots?

b. Find odds against getting 3 dots?

Solution

Let S be possible outcomes in throwing a die,

Let E be event of getting 3 dots.

Let E ' be event not getting 3 dots.

You are given that n(S) = 6, n(E) = 1 and n (*E* ') = (6 - 1) = 5

a. Odds in favor $= \frac{n(E)}{n(E')} = \frac{1}{5} = 1: 5$

Therefore, odds in favor of getting "3 dots" is $\frac{1}{5}$ or 1 : 5 .

b. the Odds against $= \frac{n(E')}{n(E)} = 5: 1 = \frac{5}{1}$

Therefore, the odds against getting "3 dots" is 5: 1 or $\frac{5}{1}$.

Exercise 8.21

Find the odds in favor of and against each event:

a. Rolling a die and getting a number less than 3.

b. Drawing a card and getting a red card.

c. Tossing two coins and getting two tails.

8.7.4 The Rules of Addition of Probability

Activity 8.11

Consider two events E_1 and E_2, what condition should be applied to the rule: $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$ and $P(E_1 \cup E_2) = P(E_1) + P(E_2)$. Discuss and verify using venn diagram?

From Activity 8.11, you have observed that, if $E_1, E_2, \ldots, E_n$ form a set of exhaustive events of a sample space S, then $E_1 \cup E_2 \cup \ldots \cup E_n = S$. Moreover, the probability of an event E, i.e., P(E) is given by

$$P(E) = \frac{\text{number of outcomes favoring E}}{\text{total of outcomes in the sampling sapce}} = \frac{n(E)}{n(S)}.$$

Having this concept, we can easily calculate probabilities of two events by making the use of the addition rule stated below.

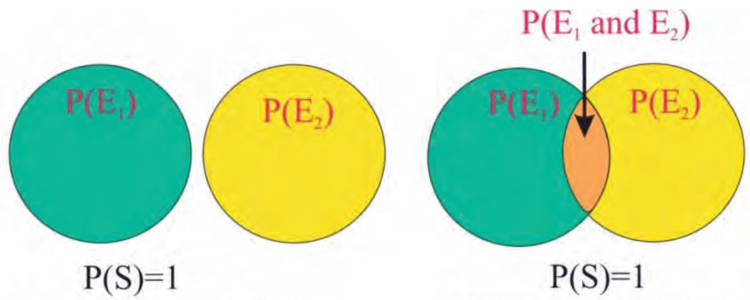
Rule 1: If E_1 and E_2 are any two events, then,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \text{ and}$$

Rule 2: If two events are mutually exclusive, (i.e., $E_1 \cap E_2 = \emptyset$) then

$$P(E_1 \cap E_2) = 0 \text{ so that } P(E_1 \cup E_2) = P(E_1) + P(E_2).$$

Moreover, you can verify this rule using venn diagram as follows



(a) Mutually exclusive events $P(E_1 \text{ or } E_2) = P(E_1) + P(E_2)$

(b) Non-mutually exclusive events $P(E_1 \text{ or } E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

Example 1

a. Find the probability of getting an odd number or 4 in one roll of a die.

b. Find the probability of getting Head or Tail in tossing a coin once.

c. A die is rolled once. Find the probability that it is even or it is divisible by 3.

Solution

a. $S = \{1, 2, 3, 4, 5, 6\}$,

Let E_1 be event of getting an odd number,

E_2 be event of getting 4;

then E_1 and E_2 are mutually exclusive events.

Thus, $P(E_1 \cup E_2) = P(E_1) + P(E_2) = \frac{3}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$.

b. Let E_1 = getting Head.

E_2 = getting Tail.

The events are mutually exclusive

Thus, $P(E_1 \text{ or } E_2) = P(E_1) + P(E_2) = \frac{1}{2} + \frac{1}{2} = 1$.

c. $S = \{1, 2, 3, 4, 5, 6\}$

Let E_1 = getting an even number = $\{2, 4, 6\}$.

E_2 = getting a number divisible by 3 = $\{3, 6\}$;

then E_1 and E_2 are not mutually exclusive, because $E_1 \cap E_2 = \{6\}$

Therefore, $P(E_1 \text{ or } E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$$= \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{2}{3}.$$

Example 2

Two dice are rolled. Find the probability of getting:

a. A sum of 8, 9, or 10

b. Doubles or a sum of 7

c. A sum greater than 9 or a sum of 12

Solution

Die 1	Die2					
	1	**2**	**3**	**4**	**5**	**6**
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

From the above table

a. $n(S) = 36$

Let E_1 = getting a sum of 8 = {(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)}.

E_2 = getting a sum of 9 = {(3, 6), (4, 5), (5, 4), (6, 3)}.

E_3 = getting a sum of 10 = {(5, 5), (6, 4), (4, 6)};

then E_1, E_2 and E_3 are mutually exclusive because $E_1 \cap E_2 = E_1 \cap E_3 = E_2 \cap E_3 = \emptyset$

Therefore, $P(E_1 \text{ or } E_2 \text{ or } E_3) = P(E_1) + P(E_2) + P(E_3)$

$$= \frac{5}{36} + \frac{4}{36} + \frac{3}{36} = \frac{12}{36} = \frac{1}{3}.$$

b. $n(S) = 36$

Let E_1 = getting Doubles = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}.

E_2 = getting a sum of 7 = {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)};

then E_1 and E_2 are mutually exclusive because $E_1 \cap E_2 = \emptyset$

Therefore, $P(E_1 \text{ or } E_2) = P(E_1) + P(E_2)$

$$= \frac{6}{36} + \frac{6}{36} = \frac{12}{36} = \frac{1}{3}.$$

c. $n(S) = 36$

Let E_1 = getting a sum greater than 9 = {(4,6), (5,5), (5,6), (6,4). (6,5), (6,6)}

E_2 = getting a sum of 12 = {(6, 6)};

then E_1 and E_2 are not mutually exclusive because $E_1 \cap E_2$ = {(6, 6)}

Therefore, $P(E_1 \text{ or } E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$$= \frac{6}{36} + \frac{1}{6} - \frac{1}{6} = \frac{1}{6}.$$

Note

The probability rules can be extended to three or more events.

1. For three mutually exclusive events E_1, E_2, and E_3,

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3)$$

2. For three events that are not mutually exclusive,

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_1 \cap E_3) + P(E_1 \cap E_2 \cap E_3).$$

Exercise 8.22

1. Two dice are rolled. Find the probability of getting:
   a. a sum of 3, 6, or 11
   b. doubles or a sum of 8
   c. a sum greater than 8 or a sum divisible by 4
2. At a convention there are 7 mathematics instructors, 5 computer science instructors, 3 statistics instructors, and 4 science instructors. If an instructor is selected, then find the probability of getting a science instructor or a math instructor.
3. A single card is drawn at random from an ordinary deck of cards. Find the probability that it is either an ace or a black card.
4. In throwing a die, consider the following events.

   E_1 = the number that shows up is odd

   E_2 = the number that shows up is prime

   E_3 = the number that shows up is less than 4

   a. Determine the event $E_1 \cap E_2$
   b. Determine the number of elements in $E_2 \cap E_3$
   c. Determine the number of elements in $E_1 \cap E_2 \cap E_3$
   d. Determine $P(E_1 \cap E_3)$
   e. Determine $P(E_1 \cup E_2 \cup E_3)$

8.7.5 The Rule of Multiplication of Probability

Independent and Dependent Events

Activity 8.12

1. What knowledge do you have about rules of addition of probability?
2. Consider two events E_1 and E_2. Discuss what condition should be applied to the rule:

$P(E_1 \text{ and } E_2) = P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$ and $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2 \backslash E_1)$, whenever $P(E_1) \neq 0$.

The multiplication rule can be used to find the probability of two or more events in sequence. It is based on the concepts of independence or dependence of events discussed earlier. Let us take a brief revision of independent and dependent events.

When the occurrence of the first event does not affect the occurrence of the second event in such a way that the probability is not changed, the events are called independent whereas when the occurrence of the first event affects the occurrence of the second event in such a way that the probability is changed, the events are called dependent.

Example 1

A jar contains 4 black and 3 white balls. You draw two balls one after the other with replacement (the second is drawn after the first is replaced). Find the probability that the first ball is black and the second ball is also black.

Solution

Let event E_1 be the first ball is black.

Let event E_2 be the second ball is black;

then, $P(E_1) = \frac{4}{7}$ and $P(E_2) = \frac{4}{7}$

$$P(E_1 \cap E_2) = P(E_1) \times P(E_2) = \frac{4}{7} \times \frac{4}{7} = \frac{16}{49}.$$

Example 2

Suppose you repeat the above experiment without replacement (the second ball is drawn without the first ball being replaced).

Solution

$P(E_1) = P$ (The first ball is black.) $= \frac{4}{7}$

If the first ball is black, then $P(E_2) = \frac{3}{6}$ (One black ball has been removed.)

If the first ball is not black, then $P(E_2) = \frac{4}{6} = \frac{2}{3}$.

Example 3

A box contains 3 red balls, 2 blue balls, and 5 white balls. A ball is selected and its colour is noted. Then it is replaced. A second ball is selected and its colour is noted. Find the probability of each of the following:

a. Selecting 2 blue balls.

b. Selecting 1 blue ball and then 1 white ball.

c. Selecting 1 red ball and then 1 blue ball.

Solution

a. Let E_1 = Selecting blue in the first draw.

E_2 = Selecting blue in the second draw.

Since, the balls are replaced after each draw, the events are independent.

$P(E_1 \cap E_2) = P(E_1) \times P(E_2) = \frac{2}{10} \times \frac{2}{10} = \frac{1}{25}$.

b. Let E_1 = Selecting blue in the first draw.

E_2 = Selecting white in the second draw.

Since, the balls are replaced after each draw, the events are independent.

$P(E_1 \cap E_2) = P(E_1) \times P(E_2) = \frac{2}{10} \times \frac{5}{10} = \frac{1}{10}$.

c. Let E_1 = Selecting red in the first draw.

E_2 = Selecting blue in the second draw.

Since, the balls are replaced after each draw, the events are independent.

$P(E_1 \cap E_2) = P(E_1) \times P(E_2) = \frac{3}{10} \times \frac{2}{10} = \frac{3}{50}$.

Exercise 8.23

1. A jar contains 6 red balls, 3 green balls, 5 white balls and 7 yellow balls. Two balls are chosen from the jar, with replacement. What is the probability that both balls chosen are green?
2. A die is rolled and a coin is tossed. Find the probability of getting a prime number on the die and a head in the coin.

Conditional Probability

Identifying dependence or independence is most important in using the multiplication rule of probability. When occurrence of one event depends on the occurrence of another event, you say the second event is conditioned by the first event. This leads into what is called conditional probability.

The **conditional probability** of an event E_2 in relationship to an event E_1 is the probability that event E_2 occurs after event E_1 has already occurred. The notation for conditional probability is $P(E_2 \backslash E_1)$. This notation does not mean that E_2 is divided by E_1; rather, it means the probability that event E_2 occurs given that event E_1 has already occurred. If the occurrence or non -occurrence of E_1 does not affect the probability of E_2, or if $\mathbf{E_1}$ and $\mathbf{E_2}$ are independent, then

$\mathbf{P(E_2 \backslash E_1) = P(E_2)}$. This defines **multiplication rule of probability**. These are:

Rule 1: When two events are independent, the probability of both occurring, denoted by P (E_1 and E_2) or P ($E_1 \cap E_2$) or P (E_1 E_2) is given by:

$$P(E_1 \cap E_2) = P(E_1) \times P(E_2)$$

Rule 2: When two events are dependent, the probability of both occurring is given by:

$$P(E_1 \cap E_2) = P(E_1) \times P(E_2 \backslash E_1), \text{ whenever } P(E_1) \neq 0$$
$$= P(E_2) \times P(E_1 \backslash E_2) \text{ whenever } P(E_2) \neq 0$$

Example 4

A bag contains 6 red, 5 blue, and 4 yellow balls. Two balls are drawn, but the first ball is drawn without replacement. Find the following:

a. P (red, then blue)

b. P (blue, then blue)

Solution

a. Let E_1 = getting red in the first draw.

E_2 = getting blue in the second draw

Since, the balls are not replaced, so events are dependent.

$$P(E_1 \cap E_2) = P(E_1) \times P(E_2 \backslash E_1) = \frac{6}{15} \times \frac{5}{14} = \frac{1}{7}.$$

b. Let E_1 = getting blue in the first draw.

E_2 = getting blue in the second draw

Since, the balls are not replaced, so events are dependent.

$$P(E_1 \cap E_2) = P(E_1) \times P(E_2 \backslash E_1) = \frac{5}{15} \times \frac{4}{14} = \frac{2}{21}.$$

Note

1. Multiplication rule 1 can be extended to three or more independent events by using the formula

$$P(E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_k) = P(E_1) \times P(E_2) \times P(E_3) \times \ldots \times P(E_k).$$

2. Multiplication rule 2 can be extended to three or more dependent events by using the formula

$$P(E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_k) = P(E_1) \times P(E_2 \backslash E_1) \times P(E_3 \backslash E_1 \cap E_2) \times \ldots \times P(E_k \backslash E_1 \cap E_2 \cap \ldots \cap E_{k-1}).$$

Example 5

A jar contains 3 red, 5 green, 2 blue and 6 yellow marbles. Three marbles are drawn one after the other. Find the probability of getting a green marble on the first draw, a yellow marble on the second draw and a red marble on the third draw; if

a. each marble is drawn, but then is replaced back before the next draw.

b. the marbles are drawn without replacement.

Solution

Let E_1 = getting green, in the first draw,

E_2 = getting yellow in the second draw,

E_3 = getting red in the third draw.

a. The marbles are replaced after each draw. The events are independent.

$$P(E_1 \cap E_2 \cap E_3) = P(E_1) \times P(E_2) \times P(E_3) = \frac{5}{16} \times \frac{6}{16} \times \frac{3}{16} = \frac{45}{2{,}048}.$$

b. The marbles are not replaced after each draw. The events are dependent.

$$P(E_1 \cap E_2 \cap E_3) = P(E_1) \times P(E_2 \backslash E_1) \times P(E_3 \backslash (E_1 \cap E_2)) = \frac{5}{16} \times \frac{6}{15} \times \frac{3}{14} = \frac{3}{112}.$$

Exercise 8.24

1. If 2 cards are selected from a standard deck of 52 cards without replacement, find these probabilities:
   a. Both are spades. b. Both are the same suit. c. Both are kings.
2. Three cards are drawn from an ordinary deck and not replaced. Find the probability of these events:
   a. Getting 3 jacks b. Getting an ace, a king, and a queen in order
   c. Getting a club, a spade, and a heart in order d. Getting 3 clubs

Sequential Events

In the previous section, you saw how to determine probability of independent or dependent events using multiplication rules of probability. It is also possible to show events that are sequential using tree diagrams and tables, and calculate probabilities from these.

Example 6

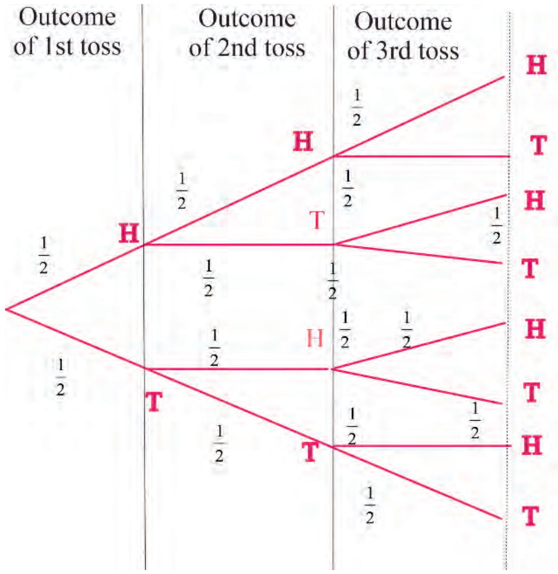
A fair coin is tossed three times. Find the probability that all outcomes will be heads.

Solution

Using the multiplication rule (HHH) = P(H) × P(H) × P(H) = $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

You can use a tree diagram and/or table to show the possible outcomes.

Using tree diagram



Squential Events	Probability of Sequential Events
HHH	
HHT	
HTH	
HTT	
THH	
THT	
TTH	
TTT	

Therefore, the probability that all outcomes are head is $\frac{1}{8}$.

Exercise 8.25

1. A fair die is rolled three times. Find the probability that all outcomes will be 6 dots.
2. If 2 cards are selected from a standard deck of 52 cards without replacement, find these probabilities:

   a. both are Aces. b. both are hearts.

8.8 Real-life Application of Probability

Probability theory is widely used in the area of studies such as statistics, finance, insurance policy, traffic signals, medical decisions, and weather forecasting. Now, you will discuss real-life applications involving probability as follows.

Example 1

A traffic light at a certain road crossing starts green at 06: 30 hours and continues to be green till 06: 32 hours and again turns green at 06: 36 hours and continues green till 06: 38 hours. This cycle is repeated throughout the day. If a person's arrival time at this crossing is random and uniform over the interval 18: 20 to 18: 35 hours, then find the probability that he has to wait the signal.

Solution

In every 6 minutes, the light remains green for 2 minutes and red for 4 minutes. So, in the interval, 18: 20 hours-18: 35 hours; i.e., 15 minutes, the light will remain green for $\frac{2}{6} \times 15 = 5$ minutes and red for 10 minutes. Thus, the probability that he has to wait at the signal $= \frac{10}{15} = \frac{2}{3} \approx 67\,\%$.

Example 2

The source of federal government revenue for a specific year is 50% from individual income taxes, 32% from social insurance payroll taxes, 10% from corporate income taxes, 3% from excise taxes and 5% from other.

If a revenue source is selected at random, what is the probability that it comes from individual or corporate income taxes?

Solution

P (revenue comes from individual or corporate income taxes) = P (revenue comes from individual) + P (revenue comes from corporate income taxes)

$$= 0.5 + 0.1$$

$$= 0.60.$$

Example 3

On New Year's Eve, the probability of a person driving while intoxicated is 0.32, the probability of a person having a driving accident is 0.09, and the probability of a person having a driving accident while intoxicated is 0.06. What is the probability of a person driving while intoxicated or having a driving accident?

Solution

P (intoxicated or accident) = P(intoxicated) + P(accident) − P (intoxicated and accident) = 0.32 + 0.09 − 0.06 = 0.35.

Exercise 8.26

1. If the probability that a person lives in an industrialized country of the world is $\frac{1}{5}$, find the probability that a person does not live in an industrialized country.
2. The top-10 selling computer software titles last year consisted of 3 for doing taxes, 5 antivirus or security programs, and 2 "other." Choose one title at random:
   a. What is the probability that it is not used for doing taxes?
   b. What is the probability that it is used for taxes or is one of the "other" programs?
3. In a group of 40 people, 10 are healthy and every person of the remaining 30 has either high blood pressure, a high level of cholesterol or both.
   a. If 15 have high blood pressure and 25 have high level of cholesterol, how many people have blood pressure and a high level of cholesterol?
   b. If a person is selected randomly from this group, what is the probability that he\she
      i) has high blood pressure only?
      ii) has high level of cholesterol only?
      iii) has high blood pressure and high level of cholesterol?
      iv) has either high pressure or high level of cholesterol?

Problem Solving

1. A company screens job application for illegal drug use at a certain stage in their hiring process. The specific test they use has a false positive rate of 2% and a false negative rate of 1%. Suppose that 5% of all their applicants are actually using illegal drug and we randomly select an applicant. Given the applicant tests positive, what is the probability that they are actually on drugs?
2. A problem is given to three students D, E, F whose respective chances of solving it are $\frac{2}{7}$, $\frac{4}{7}$ and $\frac{4}{9}$ respectively. What is the probability that the problem is solved?
3. Three bags contain 3 red, 7 black; 8 red, 2 black; and 4 red and 6 black balls, respectively. One of the bags is selected at random and a ball is drawn from it. If the ball drawn is red, find the probability of that it is drawn from the third bag.
4. A bag contains 20 marbles of equal size of which 12 are red, x is blue and the rest are white.
   a. If the probability of selecting a blue marble is $\frac{1}{4}$, find x.
   b. A marble is drawn and then replaced. A second marble is drawn. Find the probability that neither marble is red.

Summary

1. **Addition Principle of counting**

   If an operation can be performed in **m** different ways and another operation can occur in **n** different ways and the two operations are mutually exclusive, (the performance of one excludes the other) then either of the two can be performed in **m + n** ways.

2. **Multiplication Principle of counting**

   If an event can occur in **m** different ways and for every such choice another event can occur in **n** different ways, then both events can occur in the given order in **m × n** different ways.

3. If n is a natural number, then n factorial, denoted by $n!$ is defined by

   $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 \ (0! = 1)$.

4. Permutations are the number of arrangements of n objects taking r of them at a time, and denoted by nPr or P (n, r) where $nPr = P(n, r) = \frac{n!}{(n-r)!}$,

   where $0 < r \leq n$.

5. The number of combinations of n things taking r at a time is given by

   $$n\,Cr = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\,r!}.$$

6. The Binomial Theorem $(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$.

7. Probability of an event is defined as follows:

   If an experiment results in n equally likely outcomes and $m < n$ is the number of the ways favourable for event E, then $P(E) = \frac{m}{n}$.

8. Addition rule of probability

   Rule 1. If E_1 and E_2 are any two events, then,

   $$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \text{ and}$$

Rule 2. If two events are mutually exclusive; (i.e., $E_1 \cap E_2 = \emptyset$) then

$P(E_1 \cap E_2) = 0$ so that

$P(E_1 \cup E_2) = P(E_1) + P(E_2)$.

9. Multiplication rule of probability

Rule 1: When two events are independent, the probability of both occurring, denoted by $P(E_1 \text{ and } E_2)$ or $P(E_1 \cap E_2)$ or $P(E_1 \bullet E_2)$ is given by

$$P(E_1 \cap E_2) = P(E_1) \bullet P(E_2).$$

Rule 2: When two events are dependent, the probability of both occurring is given by

$$P(E_1 \cap E_2) = P(E_1) \bullet P(E_2 \backslash E_1), \text{ whenever } P(E_1) \neq 0$$

$$= P(E_2) \bullet P(E_1 \backslash E_2) \text{ whenever } P(E_2) \neq 0.$$

Review Exercise

1. A student can choose a computer project from one of three lists. The three lists contain 23, 15 and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?
2. Compute each of the following:
   a. ${}_8P_2$ b. ${}_{12}P_{10}$
   c. ${}_8C_2$ d. ${}_{12}C_{10}$
3. A committee of 7 students has to be formed from 9 boys and 5 girls. In how many ways can this be done when the committee contains:
   a. exactly three girls?
   b. at least three girls?
   c. 2 girls and 5 boys?
4. 7 boys and 6 girls are to be seated around a table. Find the number of ways that this can be done in each of the following cases:
   a. there is no restriction
   b. no girls are adjacent
   c. all girls form a single block
   d. a particular girl G is adjacent to two particular B_1 and B_2
5. A bag contains 6 red, 5 blue, and 4 yellow balls. 2 balls are drawn, but the first ball is drawn without replacement. Find the following:
   a. P (blue, then red). b. P (red, then red). c. P (yellow, then red)
6. In a pack of 52 cards, a card is drawn at random without replacement. Find the probability of drawing a queen followed by a jack.
7. Suppose that a group of 15 students contain eight boys (B) and seven girls (G). If two students are chosen randomly without replacement, find the probability that the two students chosen are both boys using multiplication rule?